I- Phénoménologie et équations; quantification des effets

Rotation

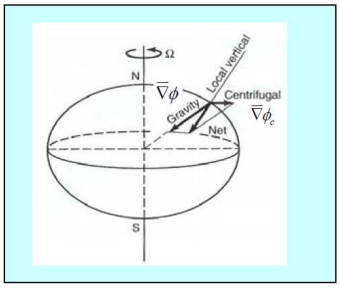
$$\rho\left(\frac{D\vec{u}}{Dt}+2\vec{\Omega}\times\vec{u}\right)=-\vec{\nabla}p+\rho\vec{\nabla}\Phi+\vec{\nabla}\cdot\overline{\overline{\tau}}$$

where $\overline{\overline{\tau}}=\mu\vec{\nabla}\vec{u}+\lambda\vec{\nabla}\cdot\vec{u}\overline{\overline{I}}$ and $\lambda=K-\frac{2}{3}\mu$ K=0 gas, otherwise also negligible.

$\Phi=\phi+\phi_c$. is the ***the geopotential***

$\phi_c=\frac{\Omega^2 r_p^{\ 2}}{2}$ is the ***centripetal potential.***

$\phi=-gz$ is the ***gravitational potential*** $(\vec{\nabla}\phi=-g\hat{k})$



Earth is an oblate spheroid (1st app), r_{eq}=6378km, r_{pol}=6357km

$\Rightarrow$ ϕ_c is weak, ie $\Phi\approx -gz$

$\Phi=0$ is called the "***geopotential surface***"

Comparaison des échelles temporelles :

$$\epsilon_T=\frac{T_{rot}}{T}=\frac{2\pi}{T\Omega}$$

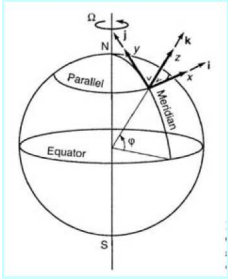
Comparaison avec temps advectif: $\epsilon=\frac{T_{adv}}{T}=\frac{2\pi U}{L\Omega}$

I- Phénoménologie et équations; quantification des effets

Rotation

	T	L	U	ϵ_T	ϵ
turbine edf		1 m	5 ms^{-1}		4 10^5
avion		5 m	100 ms^{-1}		2 10^6
tourbillon de vidange	1000 s	1 m	0.01 ms^{-1}	90	900
courant océanique		10 km	10 cms^{-1}		0.86
anticyclone		1000 km	10 ms^{-1}		0.86

Plan cartésien local :



$$2\vec{\Omega}\times\vec{u}=(2\Omega\cos\varphi w-2\Omega\sin\varphi v)\hat{i}$$
$$+(2\Omega\sin\varphi u)\hat{j}$$
$$-(2\Omega\cos\varphi u)\hat{k}$$

Define $f\equiv 2\Omega\sin\varphi$ *Coriolis parameter*

$f_*\equiv 2\Omega\cos\varphi$ *Reciprocal Coriolis parameter*

Ω = 7.29 10^{-5} rad s^{-1} f(φ=45°)≈10^{-4} $rads^{-1}$

Nombre de Rossby : $Ro=\frac{U}{fL}$

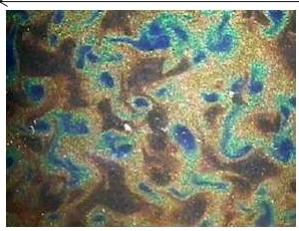
Rotation → nombre de Rossby

faible rotation (3.5 rpm)

20 cm

L=2 cm
T=1 s
Ω=0.366 s

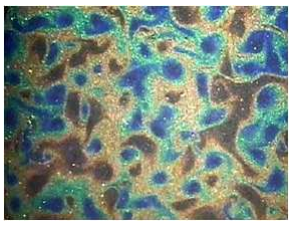
Ro=1.366



forte rotation (14 rpm)

L=2 cm
T=2 s
Ω=1.47 s

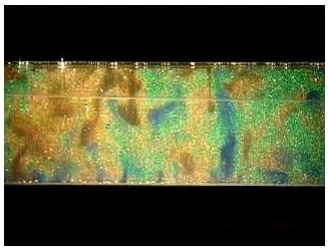
Ro=0.17

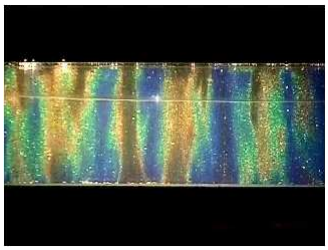


vues de dessus

6 cm

ΔT~40°C



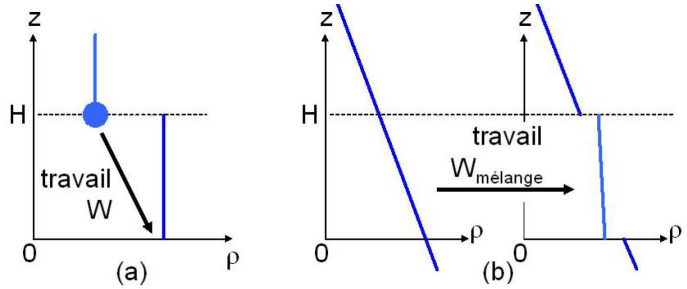


vues de côté

I- Phénoménologie et équations; quantification des effets

Approche énergétique :

Stratification



$$W_{homogeneisation} = E_{P_{apres}} - E_{P_{avant}}$$

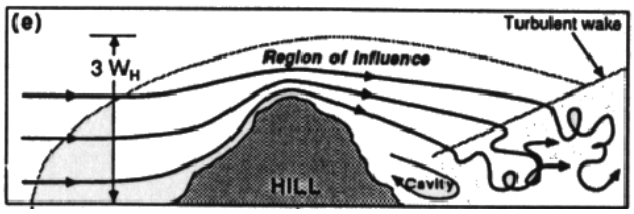
➡ Comparaison Ec et Ep ? ➡ Nombre de FROUDE

$$\sigma = \frac{\rho_0 U^2}{\Delta\rho g H}$$

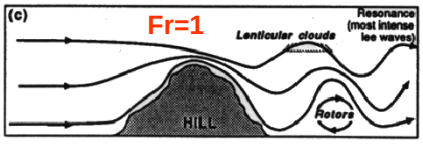
$$Fr = \sqrt{\sigma} = \sqrt{\frac{\rho_0 U^2}{\Delta\rho g H}}$$

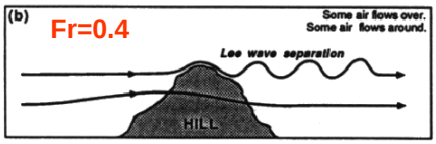
I- Phénoménologie et équations; quantification des effets

Homogène **Fr=∞**



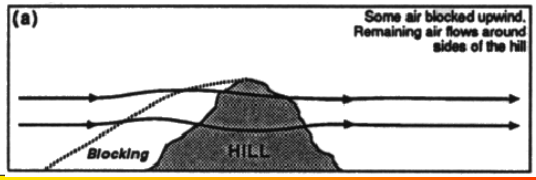
« faible » stratification





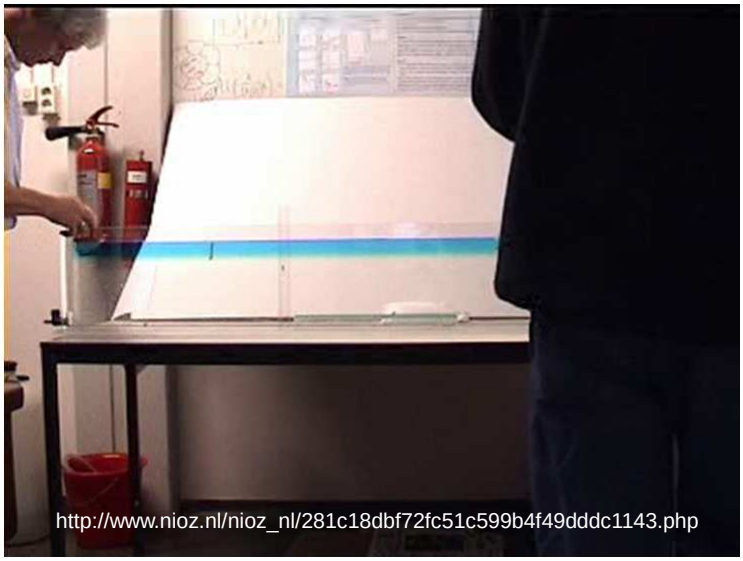
« forte » stratification

Fr=0.1



I- Phénoménologie et équations; quantification des effets

Phénomène « dead water » **Fr=1**



I- Phénoménologie et équations; stratification ?

Fréquence de Brünt -Väisälä (fréquence de flottaison)

Buoyancy force for displacement of volume V by h:

$$F_B = g\left[\rho(z+h)-\rho(z)\right]V$$

Newton's law (ma=F_B):

$$\rho(z)V\frac{d^2h}{dt^2} = g\left[\rho(z+h)-\rho(z)\right]V \quad *$$

Boussinesq ($\Delta\rho << \rho_0$), neglect $\Delta\rho$ before inertia terms

and Taylor expansion

$$\rho(z+h) = \rho(z)+\frac{d\rho}{dz}h+\ldots$$

gives:

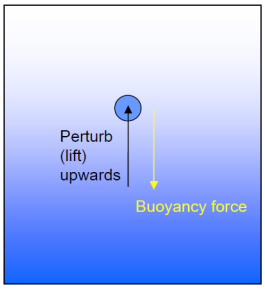
$$\rho_0\frac{d^2h}{dt^2} - g\frac{d\rho}{dz}h = 0 \quad \text{ie.,} \quad \frac{d^2h}{dt^2} - \underbrace{\frac{g}{\rho_0}\frac{d\rho}{dz}}_{\equiv N^2}h = 0$$

N is the Brunt-Väisälä frequency (rad/s) which is > 0 for dρ/dz<0!

$$N^2 = \frac{g}{\rho_0}\frac{d\rho}{dz}$$

Say, initial condition: $\frac{dh}{dt}(t=0) = W_0$ and $h(0)=0$

$$h(t) = \frac{W_0}{N}\sin Nt$$



* $b = \frac{g(\rho_o-\rho)}{\rho_o}$ vertical acceleration or buoyancy b of a volume of water at ρ surrounded by water at ρ_o

I- Phénoménologie et équations; stratification ?

Eau de mer, le plus facile… pas d'effets de compressibilité !!!

Equation of state of seawater

Density of seawater depends essentially on temperature and salinity (very little on pressure since incompressible):

$$\rho = \rho_o\left(1-\alpha(T-T_0)+\beta(S-S_o)\right)$$

$\rho_o = 1028\ \text{kg/m}^3$

$\alpha = 1.7\times10^{6}\,K^{-1}$ for $T = 283°K = 10°C$

$\beta = 7.6\times10^{-4}$ for $S_o = 35‰$

(grams of salt per kilogram of seawater or psu[1])

and since linear

$$\overline{\rho} = \rho_o\left(1-\alpha(\overline{T}-T_0)+\beta(\overline{S}-S_o)\right)$$

[1] practical salinity units: approximately 1000 times the mass of dissolved salts per unit mass of seawater.

With additional variable S, need salinity budget:

$$\frac{dS}{dt} = \kappa_s\nabla^2 S$$

Note: what diffusivity?

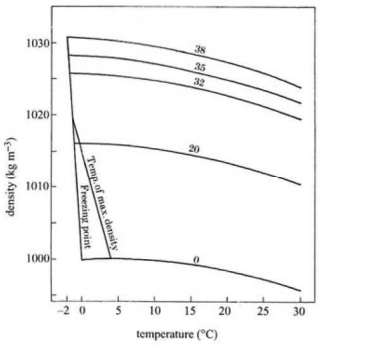


Figure 1.11. The variation of density with temperature for various values of the salinity but at atmospheric pressure. The curve on the left shows the temperature at which freezing occurs. The density of freshwater (salinity = 0 psu) is greatest at about 4 °C, but, in seawater of salinity typically ranging from 32 psu to 38 psu, the maximum density is reached at the freezing point.

(Thorpe)

I- Phénoménologie et équations; stratification ?

Water properties profiles (NA)

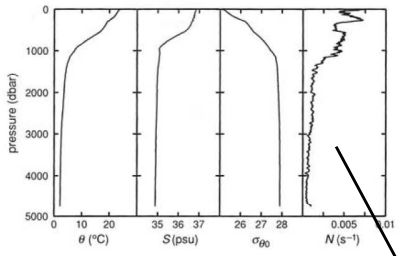


Figure 1.12. The variation of water properties with depth in the North Atlantic at 29.50′ N, 75° 42.22′ W in April 2004. From left to right: potential temperature (θ), salinity (S), sigma-theta, a measure of density (here referred to the surface, $\sigma_{\theta 0}$), and the buoyancy frequency (N) change with pressure (100 dbar ≈ 1 MPa ≈ 100 m). The profile of potential temperature exhibits a rapid decrease in temperature in the seasonal thermocline at about 200 m depth, followed by a further decrease down to about 1200 m in the main thermocline (or main pycnocline). These changes are matched in the $\sigma_{\theta 0}$ profile by a rapid increase of density in the seasonal thermocline and a more gradual increase in the main pycnocline. Relatively large values of buoyancy frequency, N, accompany these thermoclines. The small-scale structure in N possibly indicates the variations in the density gradient caused by internal waves or double diffusive convection. The near-surface and bottom mixed layers are not resolved in these profiles but may be seen in Fig. 2.4(a). (Figure kindly supplied by Dr B. A. King, National Oceanographic Centre, Southampton.)

Adiabatic heating due to pressure of water column?

Adiabatic lapse rate:

$$\Gamma_a = \frac{g\alpha' T}{c_p}$$

$\approx 1.16\times10^{-4}$ K m^{-1} at 10°C (near surface)

$\approx 1.31\times10^{-4}$ K m^{-1} at 4°C (at 4000m depth)

see Gill 1982 values of α and c_p . T is in °K.

Usually this **can be ignored**, unless water is transported vertically over very large depths or when comparing densities separated by large vertical distances. Then, **potential temperature and potential density** must be introduced.

Modèle continu ?
Modèle à 2 (N) couches ?

I- Phénoménologie et équations; stratification ?

Atmosphère… problèmes de compressibilité… mais Gaz Parfait !

Potential **temperature**: *the temperature variations due to changes in pressure (altitude) are eliminated.*

The potential temperature θ is therefore defined as the temperature that would result if a parcel of air was brought down adiabaticallv to a standard or reference pressure, taken as p_0=1000hPa.

Thus, for an isentropic process: ou $\theta = T\left(\frac{\rho_0}{\rho}\right)^{\frac{R}{MC_v}}$

$$\theta = T\left(\frac{p}{p_0}\right)^{-R/C_p} \qquad \left(\frac{T}{p^{R/C_p}} = const\right)$$

⇒ θ is the dynamically significant temperature for dry air.

Equation de la chaleur (thermodynamique)

$$\rho\frac{De}{Dt} + P(DV/Dt)/V = \delta Q/V \quad \text{càd} \quad \rho C_v\frac{DT}{Dt} - P(D\rho/Dt)/\rho = K_{TE}\Delta T$$

Equation en θ $\rho C_v\frac{D\theta}{Dt} = K_{TE}\frac{\theta}{T}\Delta T$

Intérêts? 1) Dans une transformation adiabatique, θ=cste…
2) permet une comparaison à la même pression !!!

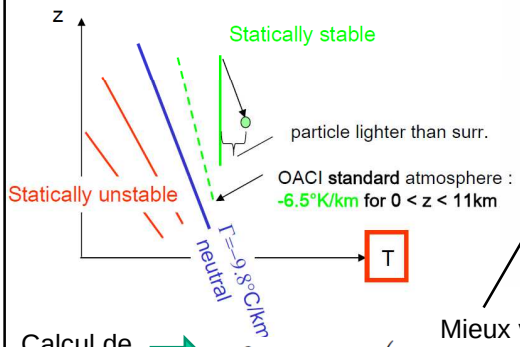
I- Phénoménologie et équations; stratification ?

Stratification des couches atmosphériques…

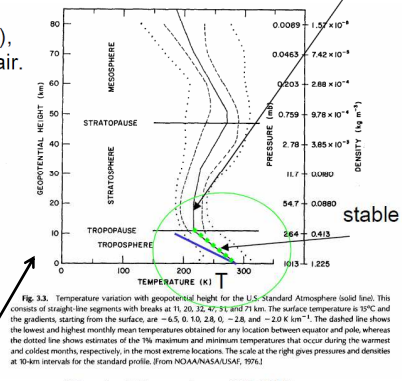
The adiabatic (vertical) temperature gradient can be shown to be (e.g. Gill 1982)

$$\frac{dT}{dz} \equiv \Gamma = -\frac{g}{C_{pd}}.$$

Γ is the (dry) adiabatic lapse rate (9.8°C / km for air),
C_{pd} is the specific heat at constant pressure for dry air.



very stable



stable

Fig. 3.3. Temperature variation with geopotential height for the U.S. Standard Atmosphere (solid line). This consists of straight-line segments with breaks at 11, 20, 32, 47, 51, and 71 km. The surface temperature is 15°C and the gradients, starting from the surface, are −6.5, 0, 1.0, 2.8, 0, −2.8, and −2.0 K km⁻¹. The dashed line shows the lowest and highest monthly mean temperatures obtained for any location between equator and pole, whereas the dotted line shows estimates of the 1% maximum and minimum temperatures that occur during the warmest and coldest months, respectively, in the most extreme locations. The scale at the right gives pressures and densities at 10-km intervals for the standard profile. [From NOAA/NASA/USAF, 1976.]

Standard Atmosphere, Gill 1982

Mieux vaut tracer les profils de θ (ou ρ_{pot})…

Calcul de N ??? ➡ $N^2 = -1/\rho_0 \left(d\rho/dz|_{milieu} - d\rho/dz|_{deplacement\ adiabatique} \right) g$

II- Equations générales et simplification

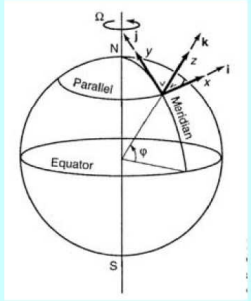
Best coordinate system : oblate spherical coordinates (for Φ =0 oblate spheroids as 1st approx.)
Next best coordinate system : spherical polar coordinates with 0.17% error taking z w.r.t. Φ=0 but for r=const.

➡ Approximation du plan f

Working coordinate system for L << r_{earth} : local Cartesian

$$\vec{\Omega} = \Omega\cos\varphi\,\hat{j} + \Omega\sin\varphi\,\hat{k}$$

$$2\vec{\Omega}\times\vec{u} = (2\Omega\cos\varphi w - 2\Omega\sin\varphi v)\hat{i} + (2\Omega\sin\varphi u)\hat{j} - (2\Omega\cos\varphi u)\hat{k}$$



Define $f \equiv 2\Omega\sin\varphi$ *Coriolis parameter*

$f_* \equiv 2\Omega\cos\varphi$ *Reciprocal Coriolis parameter*

Ω = 7.29 10^{-5} rad s^{-1}
f(φ=43.6°)=5.03 10^{-5} rad s^{-1}

f(y) ➡ Approximation du plan β : $f(y)=f_0+\beta y$

Note : f > 0 in northern hemisphere, f<0 in southern hemisphere
f_* > 0 both north and south; f_*=0 at poles.

II- Equations générales et simplification

The **momentum equations** thus becomes:

$$\rho\left(\frac{Du}{Dt}+f_*w-fv\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}$$

$$\rho\left(\frac{Dv}{Dt}+fu\right)=-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}$$

$$\rho\left(\frac{Dw}{Dt}-f_*u\right)=-\frac{\partial p}{\partial z}-\rho g+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}$$

The **Boussinesq approximation:**

Observation : density variations in the atmosphere and oceans are generally small.

$$\rho=\rho_0+\rho'(x,y,z,t),\ \rho'\ll\rho_o$$
$$p=p_0(z)+p'(x,y,z,t),\ p'\ll p_o$$

p_o,ρ_o = reference values (hydrostatic : $u=0,\ p_0(z)=P_o-\rho_o gz$

p',ρ' = fluctuations due to stratification (and/or turbulence)

II- Equations générales et simplification

For incompressible flow and Newtonian fluid, momentum equations become:

$$(\rho_o+\rho')\left(\frac{D\vec{u}}{Dt}+2\vec{\Omega}\times\vec{u}\right)=-\vec{\nabla}p-(\rho_o+\rho')g\hat{k}+\mu\vec{\nabla}^2\vec{u}$$

Since $\rho'\ll\rho_o$, ρ' ***can be neglected before the inertia term*** but NOT before g ! WHY?

$$\frac{Du}{Dt}+f_*w-fv=-\frac{1}{\rho_o}\frac{\partial p}{\partial x}+\nu\vec{\nabla}^2u$$ dividing by ρ_o

$$\frac{Dv}{Dt}+fu=-\frac{1}{\rho_o}\frac{\partial p}{\partial y}+\nu\vec{\nabla}^2v$$

$$\frac{Dw}{Dt}-f_*u=\underbrace{-\frac{1}{\rho_o}\frac{\partial p}{\partial z}-\frac{1}{\rho_o}\rho' g}_{\text{careful!}}-\frac{1}{\rho_o}\rho_o g+\nu\vec{\nabla}^2w$$ terms cancel out!

With $p=p_o(z)+p'(x,y,z,t)$ ⇨ $-\frac{1}{\rho_o}\frac{\partial p}{\partial z}=-\frac{1}{\rho_o}\frac{dp_o}{dz}-\frac{1}{\rho_o}\frac{\partial p'}{\partial z}=+\frac{1}{\rho_o}\rho_o g-\frac{1}{\rho_o}\frac{\partial p'}{\partial z}$

⇨ $\frac{Dw}{Dt}-f_*u=\underbrace{-\frac{1}{\rho_o}\frac{\partial p'}{\partial z}-\frac{\rho'}{\rho_o}g}_{\text{both terms of same order!}}+\nu\vec{\nabla}^2w$

Note that $\frac{\partial p}{\partial x}=\frac{\partial p'}{\partial x}$ and $\frac{\partial p}{\partial y}=\frac{\partial p'}{\partial y}$

II- Equations générales et simplification

We are now essentially considering the large scales (the macro β scales).

Variable	Scales	Atmospheric values	Oceanic values
x,y	L	10^5 m	10^4 m
z	H	10^3m	10^2 m
t	T	> 12h (=4 10^4s)	> 12h (=4 10^4 s)
u,v	U	10 m/s	0.1 m/s
p'	$\Delta\rho$	$0.01\rho_o$	$0.1\%\ \rho_o$

$T \sim \frac{1}{\Omega}$ ($\Omega=2\pi/24h=7.29\cdot10^{-5} rad/s$)

$U \lesssim \Omega L$ and $H \ll L$

Continuity (incompressible):

$$\underbrace{\frac{\partial u}{\partial x}}_{\frac{U}{L}}+\underbrace{\frac{\partial v}{\partial y}}_{\frac{U}{L}}+\underbrace{\frac{\partial w}{\partial z}}_{\frac{W}{H}}=0$$

Simplify?

3 choices:

(i) $\frac{W}{H} \gg \frac{U}{L} \Rightarrow \frac{\partial w}{\partial z} \approx 0 \Rightarrow w = const.$

Since bottom boundary: $w=0$, ie incompatible with $\frac{W}{H} \gg \frac{U}{L}$ assumption!

(ii) $\frac{W}{H} \ll \frac{U}{L} \Rightarrow \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \approx 0 \Rightarrow$ OK.

(iii) $\frac{W}{H} \approx \frac{U}{L} \Rightarrow$ OK.

Thus, $W \lesssim U\frac{H}{L}$ and since $H \ll L$, $W \ll U$ (1)

II- Equations générales et simplification

$$\begin{aligned}
\frac{\partial u}{\partial t}+\left(\vec{u}.\vec{\nabla}\right)(u) &= fv-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}+\nu_E\Delta u\\
\frac{\partial v}{\partial t}+\left(\vec{u}.\vec{\nabla}\right)(v) &= -fu-\frac{1}{\rho_0}\frac{\partial p'}{\partial y}+\nu_E\Delta v\\
\frac{\partial w}{\partial t}+\left(\vec{u}.\vec{\nabla}\right)(w) &= -\frac{\rho'}{\rho_0}g-\frac{1}{\rho_0}\frac{\partial p'}{\partial z}+\nu_E\Delta w\\
div(\vec{u}) &= 0\\
\frac{\partial \rho'}{\partial t}+\left(\vec{u}.\vec{\nabla}\right)(\rho')+w\frac{d\rho_0}{dz} &= \alpha_E\Delta\rho'
\end{aligned}$$

– pour les longueurs :

$x = Lx^*,\ y = Ly^*,\ z = Hz^*$

– pour les vitesses :

$u = Uu^*,\ v = Uv^*,\ w = Ww^*$

– pour le temps :

$t = Tt^*$

Et pour dimensionner P' et ρ' ?

si la rotation domine… $p' = fUL\rho_0 p^*$

si ρ' varie de Δρ suivant la verticale H… $\rho' = \Delta\rho\ \rho^*$

Pour la continuité, déjà fait

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}+\frac{\partial w^*}{\partial z^*}\frac{WL}{HU}=0 \Rightarrow \frac{WL}{HU} \leq 1$$

II- Equations générales et simplification

Pour la quantité de mouvement horizontale :

$$\underbrace{\frac{1}{fT}}_{Ro_T}\frac{\partial u^*}{\partial t^*}+\underbrace{\frac{U}{fL}}_{Ro}\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}+\frac{WL}{HU}w^*\frac{\partial u^*}{\partial z^*}\right)=v^*-\frac{\partial p^*}{\partial x^*}+\underbrace{\frac{\nu}{fL^2}}_{Ek_L}\Delta_h u^*+\underbrace{\frac{\nu}{fH^2}}_{Ek_H}\frac{\partial^2 u^*}{\partial z^{*2}}$$

$$\underbrace{\frac{1}{fT}}_{Ro_T}\frac{\partial v^*}{\partial t^*}+\underbrace{\frac{U}{fL}}_{Ro}\left(u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}+\frac{WL}{HU}w^*\frac{\partial v^*}{\partial z^*}\right)=-u^*-\frac{\partial p^*}{\partial y^*}+\underbrace{\frac{\nu}{fL^2}}_{Ek_L}\Delta_h v^*+\underbrace{\frac{\nu}{fH^2}}_{Ek_H}\frac{\partial^2 v^*}{\partial z^{*2}}$$

où Ro_T est le nombre de Rossby temporel, Ro le nombre de Rossby « classique », Ek_L le nombre d'Ekman horizontal et Ek_H le nombre d'Ekman vertical.

Pour la quantité de mouvement verticale :

$$Ro_T\underbrace{\left(\frac{WL}{HU}\right)}_{\leq 1}\underbrace{\left(\frac{H}{L}\right)^2}_{<<1}\frac{\partial w^*}{\partial t^*}+Ro\underbrace{\left(\frac{WL}{HU}\right)}_{\leq 1}\underbrace{\left(\frac{H}{L}\right)^2}_{<<1}\left(u^*\frac{\partial w^*}{\partial x^*}+v^*\frac{\partial w^*}{\partial y^*}+\frac{WL}{HU}w^*\frac{\partial u^*}{\partial z^*}\right)$$

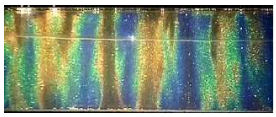
$$=-\frac{\partial p^*}{\partial z^*}-Ro\underbrace{\left(\frac{(\Delta\rho/\rho_0)gH}{U^2}\right)}_{1/Fr^2}\rho^*+\underbrace{\left(\frac{WL}{HU}\right)}_{\leq 1}\underbrace{\left(\frac{H}{L}\right)^2}_{<<1}\left(Ek_L\Delta_h w^*+Ek_H\frac{\partial^2 w^*}{\partial z^{*2}}\right)$$

II- Equations générales et simplification

Interprétation des nombres sans dimension…

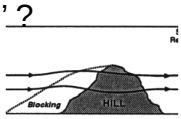
Ro
$Ro(H/L)^2$
Ro_T
Ro/Fr^2

Le cas Ro<<1
et Ro/Fr^2<<1 est très clair
(couche homogène en rotation)



Mais attention… tout repose sur le dimensionnement de P' et ρ' !!!

Exercice : sans rotation, comment dimensionner P' ?
adimensionner alors les équations…
on retrouve bien $1/Fr^2$ devant ρ*



Remarque 1 : avec Ro << 1, l'hydrostasie liée aux accélérations verticales négligeables devient encore plus facile !!! $(H/L)^2 \rightarrow Ro(H/L)^2$

Remarque 2 : si Ro/Fr^2 >> 1, on pourrait croire que ρ*=0 à l'ordre le plus bas… aucun mouvement vertical…
Il faut choisir un autre dimensionnement : $\longrightarrow \rho' \approx \rho_0 \frac{U^2}{gH}$