Supplemental Material to
"A weakly nonlinear model with exact coefficients for the fluttering and spiraling motions of buoyancy-driven bodies"

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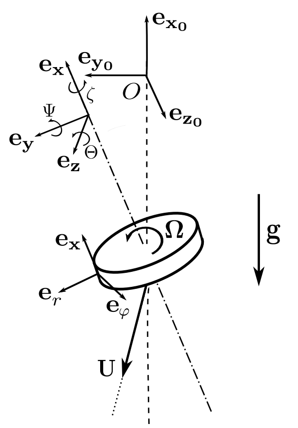

FIG. 1: Problem configuration. The disk is assumed to be initially released with its axis oriented in the vertical direction.

In this Supplemental Material we detail the weakly nonlinear analysis performed on the set of governing equations (1) that leads to the amplitude equations (3) of the paper.

Setting $\epsilon^2 = (Ar - Ar_c)/Ar_c^2$ and assuming $\epsilon \ll 1$, a Taylor expansion of all variables involved in the state vector $\mathbf{Q} = [\mathbf{V}(\mathbf{r},t), P(\mathbf{r},t), \mathbf{U}(t), \mathbf{\Omega}(t), \mathbf{\Xi}(t)]^T$ is performed up to order ϵ^3.

The full problem is solved using a system of axes with unit vectors $(\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z})$ rotating with the body (see Figure 1). Therefore, the gravity vector $\mathbf{g} = -g\mathbf{e_{x0}}$ has to be projected from the laboratory axes $(\mathbf{e_{x0}}, \mathbf{e_{y0}}, \mathbf{e_{z0}})$ onto the coordinate system $(\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z})$ and expanded in the form $\mathbf{g} = g\{\mathbf{g_0} + \epsilon\mathbf{g_1} + \epsilon^2\mathbf{g_2} + \epsilon^3\mathbf{g_3} + ...\}$. This yields

$$\mathbf{g_0} = -\mathbf{e_x},$$

$$\mathbf{g_1} = \theta_1\mathbf{e_y} - \psi_1\mathbf{e_z},$$

$$\mathbf{g_2} = \theta_2\mathbf{e_y} - \psi_2\mathbf{e_z} + \frac{1}{2}(\theta_1^2 + \psi_1^2)\mathbf{e_x} - \zeta_1(\psi_1\mathbf{e_y} + \theta_1\mathbf{e_z}),$$

$$\mathbf{g_3} = \theta_3\mathbf{e_z} - \psi_3\mathbf{e_y} - \zeta_1(\psi_2\mathbf{e_y} + \theta_2\mathbf{e_z}) - \zeta_2(\psi_1\mathbf{e_y} + \theta_1\mathbf{e_z}) - \frac{1}{2}\zeta_1^2(\theta_1\mathbf{e_y} - \psi_1\mathbf{e_z}) + (\theta_1\theta_2 + \psi_1\psi_2)\mathbf{e_x} - \frac{1}{6}(\theta_1^3\mathbf{e_y} - \psi_1^3\mathbf{e_z}) + \frac{1}{2}\theta_1^2\psi_1\mathbf{e_z},$$

where ζ_i, ψ_i and θ_i denote the coefficient of order i in the expansion of the roll, pitch and yaw angles, respectively. Straightforward symmetry considerations allow the above expressions to be significantly simplified since the odd (resp. even) coefficients involved in the expansion of ζ (resp. ψ and θ) may be shown to be zero for the antisymmetric body oscillations considered in this paper, *ie.* $\zeta_1 = \psi_2 = \theta_2 = 0$. Conversely, $\psi_1 = \theta_1 = \zeta_2 = 0$ for symmetric oscillations about the body axis.

The problem is made dimensionless by normalizing lengths with l_1, velocities with U_g, stresses with ρU_g^2 and time with l_1/U_g, l_1 and U_g having been defined in the main text. Injecting the above expansions in the system of governing equations (1) yields a nonlinear problem at zeroth order and a linear problem at each order ϵ^j for $j \geq 1$. These problems are solved for increasing j, using the finite element solver FreeFem++ [1]. The numerical details regarding mesh refinement, polynomial interpolation of velocity and pressure fields, *etc.* may be found in [2, 3].

A. Order ϵ^0: base flow

The zeroth-order or base flow is sought in the form of an axisymmetric flow associated with a steady vertical motion of the body symmetry axis. The corresponding state vector merely reads $\mathbf{Q_0} = [\mathbf{V_0}(\mathbf{r}), P_0(\mathbf{r}), -U_0\mathbf{e_x}, \mathbf{0}, \mathbf{0}]$. Using an iterative Newton method [4], the solution corresponding to the threshold $Ar = Ar_c$ is obtained by solving the steady axisymmetric version of the system (1), which in dimensionless form reads

$$\nabla \cdot \mathbf{V_0} = 0, \tag{1a}$$

$$(\mathbf{V}_0 + U_0\mathbf{x}) \cdot \nabla\mathbf{V_0} = -\nabla P_0 + Ar_c^{-1}\nabla^2\mathbf{V}_0, \tag{1b}$$

$$D_0\frac{\bar{\rho} - 1}{|\bar{\rho} - 1|}\mathbf{e_x} = \int_{\mathscr{S}} \mathbf{T_0} \cdot \mathbf{n}\mathrm{dS}, \tag{1c}$$

where $\mathbf{T_0} = -P_0\mathbf{I} + Ar_c^{-1}\mathbf{S}_0$ is the stress tensor (with $\mathbf{S}_0 = \nabla\mathbf{V}_0 +^{\mathrm{T}} \nabla\mathbf{V}_0$), and $D_0 = \dfrac{4\pi}{3}$ for both disks and oblate bubbles. The velocity $\mathbf{V_0}$ has to vanish far from the body and the no-slip (resp. shear-free) condition $\mathbf{V}_0 = U_0\mathbf{e_x}$ (resp. $\mathbf{n} \times (\mathbf{S_0} \cdot \mathbf{n}) = \mathbf{0}$) applies on the body surface for disks (resp. bubbles), $\mathbf{n}$ denoting the unit normal to this surface. The force balance (1c) defines the value of U_0, or equivalently, the relation between the Reynolds number $Re = U_0\ell_1/\nu$ and the Archimedes number $Ar = U_g\ell_1/\nu$.

B. Order ϵ^1: linear disturbances

At order 1, the linearized dimensionless equations governing the disturbances about the above base flow read

$$\nabla\cdot\mathbf{v}_1 = 0, \quad (2a)$$
$$\partial_t\mathbf{v}_1 + (\mathbf{V}_0 + U_0\mathbf{e_x})\cdot\nabla\mathbf{v}_1 + (\mathbf{v}_1 - \mathbf{w}_1)\cdot\nabla\mathbf{V_0} + \boldsymbol{\omega}_1\times\mathbf{V_0} = -\nabla p_1 + Ar_c^{-1}\nabla^2\mathbf{v}_1, \quad (2b)$$
$$M_0\frac{\bar\rho}{\chi}\{d_t\mathbf{u_1} - U_0\boldsymbol{\omega}_1\times\mathbf{e_x}\} - D_0\mathbf{g_1} = \int_{\mathscr{S}}\mathbf{t}_1\cdot\mathbf{n}\mathrm{dS}, \quad (2c)$$
$$\frac{\bar\rho}{\chi}\mathbb{I}_0\cdot d_t\boldsymbol{\omega}_1 = \int_{\mathscr{S}}\mathbf{r}\times(\mathbf{t_1}\cdot\mathbf{n})\mathrm{dS}, \quad (2d)$$
$$d_t\boldsymbol{\xi}_1 = \boldsymbol{\omega}_1, \quad (2e)$$

with $M_0 = \frac{\pi}{4}$ (resp. $\frac{\pi}{6}$) and $\mathbb{I}_0 = \frac{\pi}{64}\{2\mathbf{e_x}\mathbf{e_x} + (1+\frac{4}{3}\chi^{-2})(\mathbf{e_y}\mathbf{e_y} + \mathbf{e_z}\mathbf{e_z})\}$ (resp. $\mathbb{I}_0 = \frac{\pi}{120}\{2\mathbf{e_x}\mathbf{e_x} + (1+\chi^{-2})(\mathbf{e_y}\mathbf{e_y} + \mathbf{e_z}\mathbf{e_z})\}$) for disks (resp. oblate bubbles). These equations can be recast in the matrix form

$$\partial_t\mathscr{B}\mathbf{q}_1 + \mathscr{A}(\mathbf{Q}_0)\mathbf{q}_1 = \mathbf{0}, \quad (3)$$

where

$$\mathscr{A} = \begin{pmatrix}\mathscr{A}^f(Ar_c) & \mathscr{C}\\ \mathscr{L}(Ar_c) & \mathscr{A}^b(Ar_c,\frac{\bar\rho}{\chi})\end{pmatrix} \quad \mathscr{B} = \begin{pmatrix}\mathscr{B}^f & 0\\ 0 & \mathscr{B}^b(\frac{\bar\rho}{\chi})\end{pmatrix}$$

The matrices $\mathscr{A}^f$ and $\mathscr{B}^f$ represent the linear operators acting only upon the fluid variables $\mathbf{q}_1^f = [\mathbf{v}_1, p_1]$. Similarly, $\mathscr{A}^b$ and $\mathscr{B}^b$ are the operators corresponding to the linearized rigid body equations which only act upon the body kinematic variables $\mathbf{q}_1^b = [\mathbf{u_1}, \boldsymbol{\omega}_1, \boldsymbol{\xi}_1]$. Terms $\mathscr{C}$ and $\mathscr{L}$ ensure the coupling between the body and fluid.
Seeking solutions of (3) made of a superposition of eigenmodes for disks and oblate spheroidal bubbles was achieved in [3] and [5], respectively. As explained in the main text, we took advantage of the axisymmetry of the bodies under consideration by expressing fluid variables in the cylindrical coordinate system (x, r, φ) moving with the body and expanding them in a series of azimuthal Fourier modes of the form $e^{im\varphi}$. Since it was shown in [3] and [5] that the $m = \pm 1$ modes are the most unstable and therefore the most likely to explain the departure from a vertical path, the part of the $O(\epsilon)$ solution most relevant to the present analysis simply reads

$$\epsilon\hat{\mathbf{q}}_1 = \hat{A}^+(\tau)\left[\hat{\mathbf{q}}_{+1}^f e^{+\mathrm{i}\varphi}, \hat{\mathbf{q}}_{+1}^b\right]e^{\mathrm{i}\lambda_i t} \quad (4)$$
$$+\hat{A}^-(\tau)\left[\hat{\mathbf{q}}_{-1}^f e^{-\mathrm{i}\varphi}, \hat{\mathbf{q}}_{-1}^b\right]e^{\mathrm{i}\lambda_i t} + c.c, \quad (5)$$

where $\hat{A}^\pm(\tau)$ is the complex amplitude of the mode $\hat{\mathbf{q}}_{\pm1}$ which is linearly unstable for $Ar \geq Ar_c$.

C. Order ϵ^2: higher harmonics

At order 2, the flow is modified by higher-order harmonics which obey the system of dimensionless equations

$$\nabla\cdot\mathbf{v}_2 = 0, \quad (6a)$$
$$\partial_t\mathbf{v}_2 + (\mathbf{V}_0 + U_0\mathbf{e_x})\cdot\nabla\mathbf{v}_2 + (\mathbf{v}_2 - \mathbf{w}_2)\cdot\nabla\mathbf{V_0} + \boldsymbol{\omega}_2\times\mathbf{V_0} = -\nabla p_2 + Ar_c^{-1}\nabla^2\mathbf{v_2} - (\mathbf{v}_1 - \mathbf{w}_1)\cdot\nabla\mathbf{v}_1 - \boldsymbol{\omega}_1\times\mathbf{v}_1 - \nabla^2\mathbf{V}_0, \quad (6b)$$

$$M_0\frac{\bar\rho}{\chi}\{d_t\mathbf{u}_2 - U_0\boldsymbol{\omega}_2\times\mathbf{e_x}\} = \int_{\mathscr{S}}\mathbf{t}_2\cdot\mathbf{n}\mathrm{dS} + D_0\mathbf{g_2} - \int_{\mathscr{S}}\mathbf{S}_0\cdot\mathbf{n}\mathrm{dS} - M_0\frac{\bar\rho}{\chi}\boldsymbol{\omega}_1\times\mathbf{u}_1, \quad (6c)$$
$$\frac{\bar\rho}{\chi}\mathbb{I}_0\cdot d_t\boldsymbol{\omega}_2 = \int_{\mathscr{S}}\mathbf{r}\times(\mathbf{t}_2\cdot\mathbf{n})\mathrm{dS} - \int_{\mathscr{S}}\mathbf{r}\times(\mathbf{S}_0\cdot\mathbf{n})\mathrm{dS}, \quad (6d)$$
$$d_t\boldsymbol{\xi}_2 = \boldsymbol{\omega}_2. \quad (6e)$$

Following the notations of (3), these equations can be recast in the form

$$\partial_t\mathscr{B}\mathbf{q_2} + \mathscr{A}(\mathbf{Q}_0)\mathbf{q}_2 = \mathbf{F}_2(\mathbf{Q}_0, \mathbf{q}_1). \quad (7)$$

The forcing term $\mathbf{F_2}$ involves seven independent contributions originating from three different sources, namely the effect of small departures $Ar - Ar_c$ from the threshold Ar_c, the self-interaction of modes $\hat{\mathbf{q}}_{\pm1} + c.c.$ and the cross-interactions between modes $\hat{\mathbf{q}}_{+1} + c.c.$ and modes $\hat{\mathbf{q}}_{-1} + c.c.$ Therefore, to find the corresponding $O(\epsilon^2)$ corrections, one has to solve successively seven linear problems of the type (7). Each of them is inhomogeneous, being forced by one of the contributions in $\mathbf{F_2}$. The complete ϵ^2-order solution is obtained as the superposition of these seven partial solutions and reads formally

$$\epsilon^2\hat{\mathbf{q}}_2 = \hat{\mathbf{q}}_{(Ar-Ar_c)} + |\hat{A}^+|^2\hat{\mathbf{q}}_{(+1,+1^*)} + |\hat{A}^-|^2\hat{\mathbf{q}}_{(-1,-1^*)} + \hat{A}^{+2}\hat{\mathbf{q}}_{(+1,+1)}e^{2\mathrm{i}\varphi+2\mathrm{i}\lambda_i t} + \hat{A}^{-2}\hat{\mathbf{q}}_{(-1,-1)}e^{-2\mathrm{i}\varphi+2\mathrm{i}\lambda_i t} + \hat{A}^+\hat{A}^{-*}\hat{\mathbf{q}}_{(+1,-1^*)}e^{2\mathrm{i}\varphi} + \hat{A}^+\hat{A}^-\hat{\mathbf{q}}_{(+1,-1)}e^{2\mathrm{i}\lambda_i t} + c.c. \quad (8)$$

D. Order ϵ^3: amplitude equations

At order 3, another inhomogeneous linear system is obtained which reads

$$\partial_t\mathscr{B}\mathbf{q_3} + \mathscr{A}(\mathbf{Q}_0)\mathbf{q}_3 = \mathbf{F}_3(\mathbf{Q}_0, \mathbf{q}_1, \mathbf{q}_2). \quad (9)$$

Obviously the forcing terms gathered in $\mathbf{F_3}(\mathbf{Q_0}, \mathbf{q_1}, \mathbf{q_2})$ depend on lower-order solutions. More precisely, one has

$$\begin{aligned}
\mathbf{F}_3 = &-\partial_\tau \left[\mathbf{v}_1, 0, M_0\bar{\rho}\chi^{-1}\mathbf{u_1}, \bar{\rho}\chi^{-1}\mathbb{I}_0.\boldsymbol{\omega}_1, 0\right]^T \\
&- \left[\nabla^2\mathbf{v}_1, 0, \mathscr{F}(\mathbf{s}_1), \mathscr{M}(\mathbf{s}_1)\right]^T \\
&- \left[\mathbf{0}, 0, M_0\bar{\rho}\chi^{-1}\left(\boldsymbol{\omega}_1 \times \mathbf{u}_2 + \boldsymbol{\omega}_2 \times \mathbf{u}_1\right), \mathbf{0}, \mathbf{0}\right] \\
&- \left[\mathbf{0}, 0, \mathbf{0}, \bar{\rho}\chi^{-1}\{\boldsymbol{\omega}_1 \times (\mathbb{I}_0 \cdot \boldsymbol{\omega}_2) + \boldsymbol{\omega}_2 \times (\mathbb{I}_0 \cdot \boldsymbol{\omega}_1)\}, 0\right]^T \\
&- [(\mathbf{v}_1 - \mathbf{w}_1) \cdot \nabla\mathbf{v}_2 + (\mathbf{v}_2 - \mathbf{w}_2) \cdot \nabla\mathbf{v}_1 \\
&\qquad + \boldsymbol{\omega}_1 \times \mathbf{v}_2 + \boldsymbol{\omega}_2 \times \mathbf{v}_1, 0, -D_0\mathbf{g_3}, 0, 0]^T,
\end{aligned}$$

where $\mathscr{F}(\mathbf{s}_1) = \int_{\mathscr{S}} \mathbf{s}_1 \cdot \mathbf{n}\mathrm{dS}$ and $\mathscr{M}(\mathbf{s}_1) = \int_{\mathscr{S}} \mathbf{r} \times (\mathbf{s}_1 \cdot \mathbf{n})\mathrm{dS}$, with $\mathbf{s}_1 = \nabla\mathbf{v}_1 +^{\mathrm{T}} \nabla\mathbf{v}_1$. Among the forcing terms contained in $\mathbf{F_3}$, several contributions are proportional to $e^{\pm i\varphi \pm i\lambda_i t}$. These terms are resonant since they excite the coupled system in the directions of its unstable eigenvectors. In order to avoid singular responses of the system, we make use of the Fredholm alternative and impose that these peculiar forcing terms be orthogonal to the so-called adjoint modes [6] of the linear system. For modes corresponding to the azimuthal wavenumbers $m = \pm 1$, these adjoint modes, say $\hat{\mathbf{q}}^\dagger_{1m} = \left[\hat{\mathbf{q}}^{\dagger f}_{1m}, \hat{\mathbf{q}}^{\dagger b}_{1m}\right]^T$, obey

$$(\lambda_r - i\lambda_i)\mathscr{B}_m\hat{\mathbf{q}}^\dagger_{1m} + \mathscr{A}^\dagger_m(\mathbf{Q_0})\hat{\mathbf{q}}^\dagger_{1m} = \mathbf{0}, \qquad (10)$$

with

$$\mathscr{A}^\dagger_m = \begin{pmatrix} {}^T\mathscr{A}^f_m & {}^T\mathscr{L}_m \\ {}^T\mathscr{C} & {}^T\mathscr{A}^b_m \end{pmatrix} \quad \mathscr{B}_m = \begin{pmatrix} \mathscr{B}^f & 0 \\ 0 & \mathscr{B}^b_m \end{pmatrix}.$$

The physical meaning of these adjoint modes has been extensively discussed in the recent literature [4, 6, 7]. Proceeding with the above compatibility constraint, we finally obtain the set of amplitude equations (equation (3) in the main text) for the $O(\epsilon)$ amplitudes $\hat{A}^\pm$ in the generic form

$$d_t\hat{A}^\pm = \sigma(Ar - Ar_c)\hat{A}^\pm - \mu\hat{A}^\pm|\hat{A}^\pm|^2 - \nu\hat{A}^\pm|\hat{A}^\mp|^2, \quad (11)$$

Once the coefficients of (11) are determined for a given set of $\bar{\rho}, \chi$ and Ar_c through the procedure described above, the weakly nonlinear evolution of the corresponding physical system can be studied.

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