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Instabilities and transition in three-dimensional flows with rotation

SLOSHING MODES AND SINGULAR INERTIAL MODES IN A CYLINDRICAL TANK ROTATING AROUND ITS AXIS

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1 Introduction

The problem of sloshing of a liquid in a partially filled tank is relevant to many fields in engineering. The particular case where the tank is rotating is relevant, for instance, to the fuel tanks of satellites and spatial launchers. It is known that the fluid motion occurring in these tanks can have a destabilizing effect on the motion of the vehicle, so a prediction of the properties of these motions is highly desirable. The problem is also relevant to the tides in a closed lake or internal basin.

Here we consider the case of a partially filled cylindrical tank rotating around this axis. It is well known that in this case, a steady solution corresponds to a solid-body rotation of the liquid with a parabolic free surface. Previous investigations [3, 4] mostly studied the modes and stability of small oscillations in the inviscid case. A striking result is that the nature of the problem and of its solutions sharply varies with the ratio of the relative frequency of the eigenmode (i.e. with respect to the rotating frame) to twice the rotation rate. When this parameter is larger than 2 or smaller than -2, the problem is elliptic and admits well-defined eigenmodes which do not penetrate deeply into the fluids. On the other hand, when this parameter is within the interval $[-2, 2]$, the problem is hyperbolic, and nothing guaranties the existence of regular eigenmodes. Regular eigenmodes were actually identified in this range in particular cases : they have a different structure, more akin to the tidal waves first investigated in the shallow-water theory of Lord Kelvin. The works dealing with the viscous problem [1] seem to have been limited to the case of axisymmetric waves.

In this work we investigate the problem with three methods : a shallow-water theory, a semi-analytical method where the solutions are expanded in series of special functions, and a finite-difference solution of the viscous problem. The results are compared with previous works. A striking result is the discovery, in the hyperbolic case (as defined above), of a new set of viscous eigenmodes with a pecular structure, akin to the singular inertial modes found in other geometries such as spherical shells [5] or the atmosphere of rotating stars [2].

2 Base flow, Stability equations and resolution methods

We consider a cylindrical recipient partially filled with water and rotating around its axis. We note R the radius of the container, Ω the rotation rate and g the acceleration of gravity. It is a classical exercise to demonstrate that the free surface takes, at equilibrium, a parabolic shape defined by

$$z = h(r) \equiv h_0 + \frac{\Omega^2 r^2}{2g} - \frac{\Omega^2 R^2}{4g}, \tag{1}$$

where h_0 is the average depth of the water ; more precisely the depth that the same volume of water would fill in the recipient if the rotation was stopped. We investigate the linear stability properties of this flow by considering eigenmodes with the form $\hat{u}(r,z)e^{im\theta}e^{\sigma t}$, where r, θ, z are cylindrical coordinates, m is an azimuthal wavenumber and $\sigma = \sigma_r + i\sigma_i$ a complex frequency. These eigenmodes have to be computed as nontrivial solutions of an eigenvalue problem which can be written in the following symbolic form :

$$\mathcal{A}\hat{u} = \sigma \mathcal{B}\hat{u}. \tag{2}$$

Three methods of resolutions are used to tackle this problem. The first method is a shallow-water theory which allows to reduce the problem to a single, second-order equation which turns out to have analytical solution in terms of hyperbolic functions. This method is by nature limited to the case where the aspect ratio $\chi = h_0/R$ of the tank is small (but viscosity may be included with an ad-hoc modeling).

The second method, which is also limited to the inviscid case, consists of finding semi-analytical solutions expanded over a basis of functions with separable variables, with the following form (for the pressure component):

$$p(r,z) = \Sigma_{j=1}^{\infty} X_j \frac{\cosh(k_j z)}{\cosh(k_j h_0)} J_m(\xi_j r), \tag{3}$$

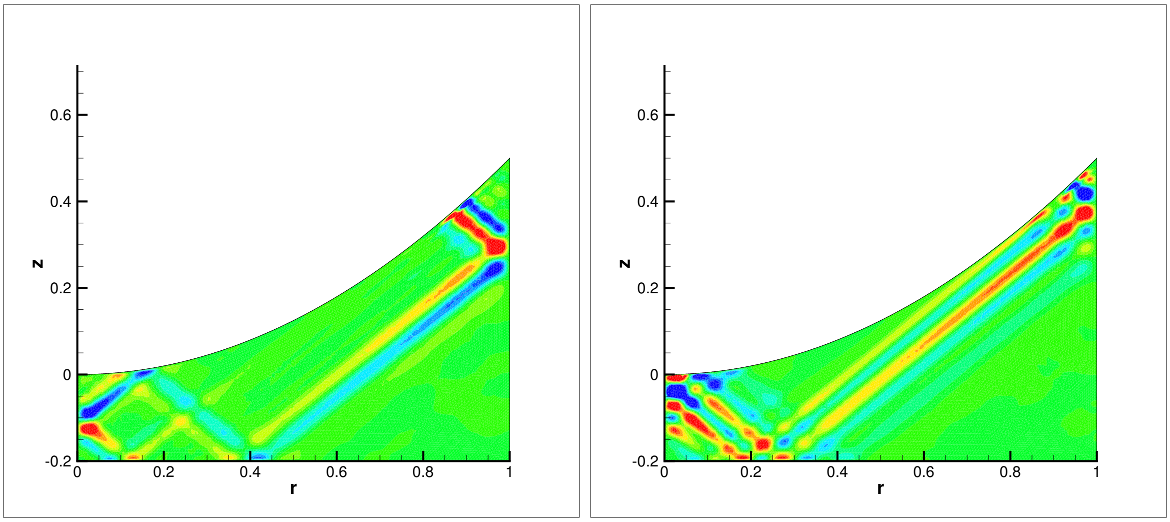

Figure 1: Exemples of singular eigenmodes in a tank with a parabolic surface

Where J_m is a Bessel function, k_j, ξ_j are vertical and radial "wavenumbers" which have to be taken among a discrete set of characteristic values, and X_j are amplitude coefficients to be solved for. Such an expansion automatically satisfies the boundary conditions at the bottom and outer boundary. Truncation of this series to a finite N and imposition of the free-surface condition at N points leads system of equations for the N amplitudes X_j, which admits nontrivial solutions for a discrete set of eigenvalues.

The third method consists of discretizing the set of partial differential equations formed by the eigenvalue problem with a finite element methods. We do this by means of the software FreeFem++. This method allows to consider the general, viscous problem.

3 Results

Before considering the general case, we consider three situations where a comparison between our three methods of resolution (and with previous work) is possible. The first particular case corresponds to weak rotation. In this case, the eigenmodes belong to two distinct sets. The first ones are "pure sloshing modes" identical to those existing in a non-rotating container. The second ones are "pure inertial modes" which are identical to those existing in a cylindrical container with a flat, rigid surface. The second particular case corresponds to weak aspect ratio h_0/R and allows use of the shallow-water approximation. The third particular case corresponds to large aspect ratio h_0/R. In this case inviscid solutions can be found in closed form [3]

We then consider the general, viscous case. In addition to the modes already identified in the litterature, we evidence a new set of eigenmode solutions with a singular structure, akin to the singuliar modes existing in spherical shells [5] or the atmosphere of rotating stars [2]. The structure of these modes is concentrated along "rays" forming a closed pattern in the geometry. Figure 1 shows sample specimens of such modes.

References

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