Supplement to
The Rotating Polygon Instability of a Swirling Free Surface Flow

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DERIVATION OF EQ. (16) AND COMPARISON WITH EXPERIMENTS

In this short document, we provide additional details and justifications regarding the argument we used to relate the parameters of the model to the rotation frequency $f = \Omega_{plate}/(2\pi)$ of the plate, in order to be able to compare the prediction of the model to experimental results.

The influx of angular momentum, i.e. the torque exerted on the fluid by the surfaces, both fixed and rotating is

$$M = \rho \int_A \mathrm{d}A \;\; r\,\sigma(r) \tag{1}$$

where the integral is over the entire solid boundary area A, i.e., the rotating bottom plate and the fixed cylinder wall, and $\sigma(r)$ is the shear stress, which we count positive if it forces the fluid in the counterclockwise direction. Here $V_{plate}(r) = \Omega_{plate} r$ and $U(r) = \frac{\Gamma}{2\pi r}$ are the azimuthal velocities of the plate and the flow, respectively.

We now need to estimate the wall shear stress $\sigma(r)$ in (1). In rotating systems, there are several types of boundary layers, each with their different scaling on Reynolds number. However, because the Reynolds number is large, we expect the fluid viscosity to be absent in our expression, an assumption supported by the experiments of [1]. So we are left with rather few quantities that must together determine σ. Those are the boundary layer thickness, the constant fluid density ρ and the velocity difference between the solid wall and the bulk flow, say ΔU. On dimensional grounds, the boundary layer thickness drops out, and we must have a relation of the form

$$\sigma \sim \rho |\Delta U| \Delta U \tag{2}$$

These assumptions are discussed in further detail in [2], §42-44, together with turbulent boundary layers and the well-known logarithmic velocity profile and are equivalent in this case to Prandtl's mixing length theory [3]. We assume, for simplicity, that (2) applies to both solid boundaries, i.e., the bottom plate and the side wall, with the same constant of proportionality.

Now, for a stationary flow we must have $M = 0$, i.e., the drag from the cylinder walls must balance the thrust from the rotating bottom plate, which, using (1) and (2), and cancelling a common factor becomes,

$$R^2 \zeta\, U(R)^2 = \int_\xi^R r^2 \mathrm{d}r \; |V_{plate}(r) - U(r)| \, (V_{plate}(r) - U(r)) \tag{3}$$

Inserting the definitions of $V_{plate}(r)$ and $U(r)$ we get equation (16) from the paper:

$$\int_\xi^R \mathrm{d}r \left(\frac{r^2}{x^2} - 1\right) \left|\frac{r^2}{x^2} - 1\right| = \zeta, \tag{4}$$

where x is the location where the angular velocity of the potential flow matches that of the plate, defined by $\Omega_{plate} x = \Gamma/(2\pi x)$. Along with Eq. (2-3) of the paper, this equation gives a relation between the flow parameters, in particular the circulation Γ and the radius of the "dry" region ξ, as function of the bottom plate rotation rate Ω_{plate}. As an illustration, these two latter relations are plotted with black lines in Fig. 1 (with a convenient non-dimensionalization), for the case $H/R = 0.276$. The relation between ξ and Ω_{plate}, plotted in the second plot, gives the relation between Fig. 2-3 of the paper (parametrized by ξ/R) and Fig. 4 (parametrized by $f = \Omega_{plate}/2\pi$).

Note that, owing to the absolute values in Eqs. (2) and (3), the friction at the bottom is a source of angular momentum at all r, only if the plate is rotating faster than the flow everywhere, i.e., $V_{plate}(r) > U(r)$, and this condition is fulfilled for $x < \xi$. On the other hand, if the rotation rate of the plate is not large enough, for instance, for $\Omega_{plate}\sqrt{R/g} < 2.32$ in the case plotted in Fig. 1, the model predicts the existence of an inner zone (defined by $\xi < r < x$) where the potential flow is rotating faster than the plate and the friction in that region slows down the fluid. This is not in agreement with observations and indicates that the simplistic modelling in terms of a pure potential vortex does not work all the way in to $r = \xi$. Presumably an inner layer can form where the velocity field switches to something closer to a rigid rotation. We are currently working on extending our model in this direction. As a first attempt, we may simply ignore the effect of this inner zone by applying the friction law (4)

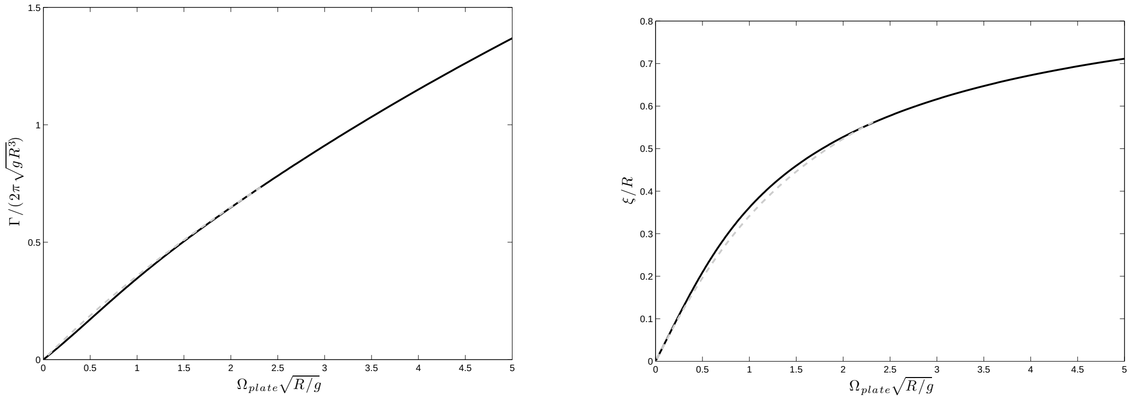

Figure 1: Circulation Γ (left) and radius of "dry" region ξ (right) predicted by the angular momentum balance argument, for the case $H/R = 0.276$. Black line: model used in the main paper. Grey dashed line: model omitting the effect of central regions rotating faster than the plate, using (5).

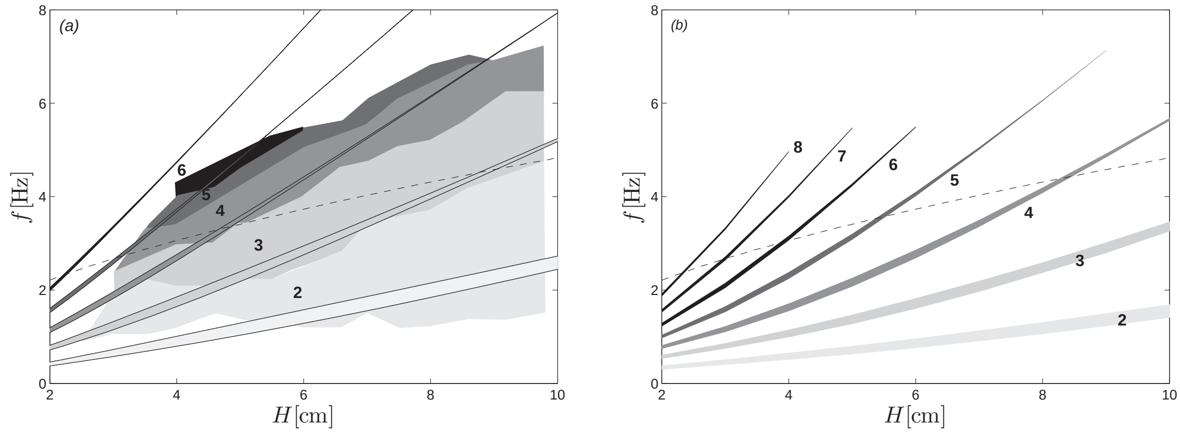

Figure 2: Phase diagram from Fig. 4 in the paper, showing the state as function of filing height H in cm and the bottom rotation rate f in Hz corresponding to a cylinder radius of $R = 14.5$ cm. The shades of grey (going from bright grey to black) indicate polygons with $m = 2, 3, 4, 5, 6$. Below the dashed line, the condition $U(r) < V_{plate}(r)$ is violated near the point A ($r = \xi$) in Fig. 1 of the paper. (a) Regions of Instability for the 2D. (b) Regions of instability from the full linearized flow.

only in the regions where the plate is rotating faster than the flow (defined by $r > x$). This would change (4) to

$$\int_x^R \mathrm{d}r \left(\frac{r^2}{x^2} - 1\right)^2 = \zeta. \tag{5}$$

The circulation Γ and the radius of the "dry" region ξ predicted by this modified relation are plotted in Fig. 1 with grey, dashed lines. As can be seen, the results remain very close to the previous ones, and the circulation Γ is only very slightly higher than the one predicted by the previous relation.

In Fig. 2a and Fig. 2b, we reproduce the "phase diagrams" from Figures 4a and 4b of the main paper, except that we have inserted a dashed line below which the model predicts an inner region rotating faster than the plate, and in which such small corrections to the results should be introduced. When comparing to the experimental data, it should also be kept in mind that our model only works for the "dry" polygons where a central part of the plate remains dry. It is not clear from the experimental data, precisely in which region this is fulfilled, but most of the ellipses ($m = 2$) and part of the triangles are actually "wet" so again the lower part of the diagram cannot strictly speaking be compared to our theory.

VIDEO OF SPIN-DOWN IN LIQUID NITROGEN

As explained in the text, we have observed the same basic instabilities when spinning down liquid Nitrogen in a hot pot. We place a kitchen pot on a hot stove and when the pot is sufficiently hot, we pour in a layer of liquid Nitrogen and stir rapidly with a wooden spoon.

Keeping the temperature so high that the Nitrogen undergoes film boiling at the solid boundary ensures that there is a thin gas layer between the fluid and the pot, which acts like a lubricant and allows the liquid to flow almost unimpeded. For a pot with a diameter of 20 cm and a layer of around 5 cm of fluid, It takes around a minute for the fluid to spin down from the violently turbulent, but basically axis-symmetric, initial state reminiscent of a potential vortex. In this process we observe a series of polygons, starting from high m and evolving toward lower m until the fluid layer has become too thin due to evaporation of too slow due to friction. Movie 1 shows the appearance of a hexagon and its gradual transition to a square. Movie 2 shows the the appearance of a square and its transition to a triangle. The movies are made with 240 pictures pr. sec, i.e. slowed down by a factor of 8. Note that that the surface waves near the edges of the pot are clearly visible.

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