Global linear stability analysis of the wake and path of buoyancy-driven discs and thin cylinders

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The stability of the vertical path of a gravity- or buoyancy-driven disc of arbitrary thickness falling or rising in a viscous fluid is investigated numerically in the framework of global linear stability. The disc is allowed to translate and rotate arbitrarily and the stability analysis is carried out on the fully coupled system obtained by linearizing the Navier-Stokes equations for the fluid and Newton’s equations for the body. Three discs with different diameter-to-thickness ratios are considered: one is assumed to be infinitely thin, the other two are selected as archetypes of thin and thick axisymmetric bodies, respectively. The analysis spans the whole range of body-to-fluid inertia ratios and considers Reynolds numbers (based on the fall/rise velocity and body diameter) up to 350. It reveals that four unstable modes with an azimuthal wavenumber \(m = \pm 1\) exist in each case. Three of these modes result from a Hopf bifurcation while the fourth is associated with a stationary bifurcation. Varying the body-to-fluid inertia ratio yields rich and complex stability diagrams with several branch crossings resulting in frequency jumps; destabilization/restabilization sequences are also found to take place in some subdomains. The spatial structure of the unstable modes is also examined. Analyzing the differences between their real and imaginary parts (which virtually correspond to two different instants of time in the dynamics of a given mode) allows us to assess qualitatively the strength of the mutual coupling between the body and fluid. Comparisons with available computational and experimental data reveal a close agreement whatever the body aspect ratio. Qualitative and quantitative differences between present predictions and known results for wake instability past a fixed disc enlighten the fact that the first non-vertical regimes generally result from an intrinsic coupling between the body and fluid and not merely from the instability of the sole wake.

Key Words: Instability, wakes, flow-structure interactions.

1. Introduction

Understanding and predicting the path of bodies of arbitrary shape falling/rising under gravity/buoyancy in an infinite fluid medium has ever been an important concern in Mechanics, as testified by Leonardo’s drawings related to the spiraling motion of rising bubbles (Prosperetti et al. 2003; Prosperetti 2004). The general equations governing the motion of a body of arbitrary geometry in an inviscid fluid at rest at infinity were established by Kirchhoff (Kirchhoff 1869) and their solutions promptly studied in a variety of situations by Thomson and Tait (Lamb 1932). However, it was already clear to
Maxwell (1853) that viscosity played a key role in the selection of the style of path daily observed in many situations, such as paper strips falling in air, seeds released from trees, coins falling in water, etc; see Ern et al. (2012) for a more exhaustive historical perspective of the subject. Nevertheless, despite this uninterrupted interest, robust experimental data were essentially made available within the last half-century, especially for thin discs (Willmarth et al. 1964; Field et al. 1997).

The topic has received growing attention during the last decade for two main reasons. One is of course its physical relevance and importance in many up-to-date applications such as in meteorology, ecology, bio-inspired flight, aeronautics and multiphase flows to mention just a few. The other is the growing maturity of modern observation techniques, such as high-speed video cameras and particle image velocimetry, and of three-dimensional direct numerical simulation (DNS). Thanks to advances in these techniques, detailed experiments in which the three-dimensional characteristics of the body displacement (including rotations) are accurately determined as a function of the control parameters have been carried out, especially for discs of various aspect ratios (Fernandes et al. 2007) and spheres (Horowitz & Williamson 2010). Some of these studies also included a detailed exploration of the wake structure and vortex shedding process (Ern et al. 2007; Horowitz & Williamson 2010; Zhong et al. 2011, 2013). DNS has also become a powerful tool to explore the various non-vertical regimes of axisymmetric bodies such as freely rising spheroidal bubbles (Mougin & Magnaudet 2002b), falling/rising spheres (Jenny, Dusek & Bouchet 2004) and discs (Auguste, Magnaudet & Fabre 2013; Chrust, Bouchet & Dusek 2013), and determine the transitions between them, especially in the presence of strongly nonlinear effects.

Nevertheless, despite its many merits, DNS does not give a detailed access to several important aspects of the phenomena that take place near the first thresholds of the instability, such as the spatial structure of each unstable mode or the reasons that sometimes make the characteristics of path deviations (especially their geometry and frequency) change dramatically when the body geometry is slightly modified, nor does it allow the role of these various modes to be easily disentangled. Such information is better obtained by performing systematic linear or weakly nonlinear global stability analyses. This approach is also the right tool for performing parametric studies of the neutral curves corresponding to destabilization of the base flow, as DNS is very time-consuming under near-neutral conditions where the temporal transients are extremely long. However it is only very recently that this type of approach started to be applied to this class of problems. A quasi-static theory (QST) was developed by Fabre, Assemat & Magnaudet (2011) to analyze the loss of stability of the vertical path of two-dimensional bodies, such as plates and rods, in the limit where their inertia is much larger than that of the surrounding fluid, which leads to a clear separation of time scales between the body, whose deviations from the vertical path can only be ‘slow’, and the fluid. The same group (Assemat, Fabre & Magnaudet 2012) then removed that restriction on the inertia ratio and carried out a systematic linear stability analysis for the same family of bodies. Regarding three-dimensional bodies, the only work to date seems to be the weakly nonlinear analysis recently performed by Fabre, Tchoufag & Magnaudet (2012) to predict the onset and geometrical characteristics of the steady oblique path of spheres and discs observed under certain conditions, e.g. Veldhuis & Biesheuvel (2007) and Horowitz & Williamson (2010) for spheres. Interestingly, the bifurcation threshold and the characteristics of this mode were shown to depend on the body geometry but not of its inertia.

As is now widely recognized, the presence of vorticity in the flow is at the root of path instability (as far as the body is released in such a way that no torque is created by the initial conditions), making the understanding of wake instability past the body held
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fixed an important prerequisite. However this does not imply that the body dynamics are slaved to the wake: some body paths clearly mirror the characteristics of wake modes, while other do not. Similarly, the thresholds at which these non-vertical paths set in and their oscillation frequencies often differ dramatically from those of wake instability (Alben 2008; Assemat et al. 2012; Auguste et al. 2013). These observations make it clear that any stability analysis of this class of problems must consider the body+fluid system as fully coupled, even though it may be concluded at the very end that the couplings are weak in some limit cases.

This is the essence of the present work in which we perform a systematic linear stability analysis (LSA) of the entire system for a class of bodies corresponding to disks of various thicknesses which may also be thought of as thin circular cylinders. In this analysis, the hydrodynamic loads experienced by the body determine its translation and rotation, while the latter dictate the boundary condition at the body surface and modify the fluid momentum balance through additional pseudo-forces. The organization of the paper is as follows. We first describe in section 2 the geometrical setup, control parameters, general governing equations and detail the LSA approach, especially the formulation of the generalized eigenvalue problem; the various operators involved in this problem and some important properties of the eigenmodes up to an azimuthal wavenumber $m = \pm 2$ are made explicit in Appendix B. The next three sections are devoted to the discussion of the results obtained for Reynolds numbers (based on the body diameter and rise/fall velocity) up to 350 for an infinitely thin disc and two discs (or thin cylinders) with diameter-to-thickness ratios of 10 and 3, respectively. The latter two geometries are selected because available experiments (Fernandes et al. 2007) suggest that they may be considered as typical of the contrasting behaviours displayed by ‘thin’ and ‘thick’ bodies, respectively. Each of these sections discusses the contents of the stability diagrams provided by the LSA as a function of the body-to-fluid inertia ratio, the nature of the corresponding unstable modes and their spatial characteristics from which we show that it is possible to infer important characteristics of the body dynamics in the fully nonlinear regime. As far as possible, results of the LSA are compared with available DNS and experimental data.

The paper ends with section 6 where results obtained for the three bodies are compared and physical mechanisms that make their dynamics strikingly different, especially in the limit of low inertia ratios, are discussed.

2. Problem formulation and methodological approach

2.1. Control parameters and geometrical configuration

We consider a three-dimensional body with uniform density and cylindrical geometry falling or rising freely under gravity in an unbounded fluid at rest at infinity. In what follows, the body which has a thickness $h$, diameter $d$ and density $\rho_b$, will be termed a ‘disc’ whatever its thickness. This body moves with an instantaneous velocity whose translational and rotational components are $\mathbf{U}(t)$ and $\mathbf{\Omega}(t)$, respectively. The surrounding medium is a Newtonian fluid of kinematic viscosity $\nu$ and density $\rho$.

The problem is entirely characterized by three dimensionless parameters for which several choices are possible (Ern et al. (2012)). The first of these is unambiguously the geometrical aspect ratio $\chi = d/h$. As a second parameter, we may choose either the body-to-fluid density ratio $\bar{\rho} = \rho_b/\rho$, or equivalently some inertia ratio $I^*$ involving the disc’s moment of inertia; the selected definition of $I^*$ will be specified later. Note that, gravity acting downwards, ‘heavy’ discs such that $\bar{\rho} > 1$ fall whereas ‘light’ discs with $\bar{\rho} < 1$ rise. The
third parameter is a ‘Reynolds’ number built upon the disc diameter, the viscosity, and a velocity scale. Several choices are possible for the latter, as the actual disc velocity is usually not known beforehand. A possibility is to use the gravitational velocity $U_g = \sqrt{\frac{2}{\rho - 1}gh}$, yielding the so-called Archimedes number $Ar = \frac{3}{32} \frac{U_g d}{\nu}$. Here, we will generally make use of the velocity $U_0$ of the disc in the ‘base state’ corresponding to the steady vertical broadside motion, which yields a ‘nominal’ Reynolds number defined as $Re = \frac{U_0 d}{\nu}$. This velocity scale coincides with the actual disc’s velocity as long as the deviations with respect to this base state are small, which is consistent with the linear study conducted herein. The relation between $Re$ and $Ar$ will be made explicit in section 2.4.

2.2. Governing equations

We first define two frames of reference sketched in Figure 1: the absolute or laboratory frame $(O, x_0, y_0, z_0)$ and the relative or body frame $(C, x, y, z)$. Fabre et al. (2011) and Assemat et al. (2012) also used a third ‘aerodynamic’ frame and showed the advantages offered by each of these three frames in the discussion of results. In the absolute frame of reference, the position of any material point of the disc may be characterized through its distance vector $r$ from the disc’s centre of inertia $C$ and the three angles $(\zeta, \Theta_y, \Theta_z)$ of the roll/pitch/yaw system measuring rotations about $x$, $y$ and $z$, respectively. With these definitions one may introduce the vector $\Xi = (\zeta, \Theta_y, \Theta_z)$ and, for small rotations, the rotation rate $\Omega = \frac{d\Xi}{dt}$ follows directly.

The flow around the disc of volume $V = \frac{4}{3} \pi d^3 h$, mass $M = \rho_0 V$ and moment of inertia tensor $\mathbb{I} = I_1 \mathbf{xx} + I_2 (\mathbf{yy} + \mathbf{zz})$ with $I_1 = \frac{1}{8} Md^2$ and $I_2 = \frac{1}{16} Md^2 (1 + \frac{4}{3\chi^2})$ is described by the incompressible Navier-Stokes equations. The system of equations governing the fluid-disc dynamics is fully coupled through the fluid forces and torques acting on the disc’s surface $\mathcal{S}$ on the one hand and the no-slip boundary condition on $\mathcal{S}$ imposed to the flow by the moving disc on the other hand. We express the equations in the absolute frame, but with axes rotating with the disc, following (Mougin & Magnaudet 2002a).
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Figure 2: Sketch of the computational domain. Gravity $\mathbf{g}$ is oriented towards the left (resp. right) for falling (resp. rising) discs.

The above equations form a coupled system governing the evolution of the state vector $\mathbf{Q} = \{\mathbf{Q}_f', \mathbf{Q}_b\}$, where $\mathbf{Q}_f' = \{\mathbf{V}(r,t), P(r,t)\}$ contains the quantities associated with the fluid and $\mathbf{Q}_b = \{\mathbf{U}(t), \Omega(t), \mathbf{E}(t)\}$ gathers the degrees of freedom associated with the body kinematics. The latter are expressed in a moving Cartesian basis ($\mathbf{x}, \mathbf{y}, \mathbf{z}$) using straightforward transformations involving the roll/pitch/yaw angles (yielding for instance disc velocity components $(U_z, U_y, U_z)$). On the other hand, quantities associated with the
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fluid are projected onto the local cylindrical basis \((e_r, e_\phi, x)\) (see Figure 1), yielding fluid velocity components \((V_r, V_\phi, V_x)\).

To perform the LSA of the problem, the flow is classically split into a base flow plus a disturbance in the form \( \mathcal{Q} = \mathcal{Q}_0 + \epsilon \mathcal{Q} \) \((\epsilon \ll 1)\). Introducing the above ansatz in (2.5), a zeroth-order non-linear problem and a first-order linear problem are obtained. Thanks to an azimuthal Fourier expansion of the disturbance (see below) and to symmetries detailed later, the fluid quantities only have to be computed in a two-dimensional domain corresponding to a meridional half-plane, as sketched in Figure 2. This domain is discretized via a Delaunay-Voronoi algorithm which generates triangular elements. The dashed lines shown in Figure 2 mark off the regions where a local refinement is applied to capture properly the boundary layer and the near-wake.

The zeroth- and first-order problems are discretized through a finite element method using the FreeFem++ software. For that purpose, a weak formulation involving \(P_2\) (quadratic) elements for each component of the velocity field and \(P_1\) (linear) elements for the pressure field is set up. This formulation takes account of the no-slip condition on the disc surface and makes use of a ‘zero-traction’ condition \(-\mathcal{P}_x + \rho \nu \cdot \nabla V = 0\) on the domain outlet. The nonlinear problem for the base flow is solved using a Newton method. Convergence is said to be reached when the \(L^2\)-norm of the velocity variations evaluated over the entire computational domain becomes less than \(10^{-11}\).

The linear problem for the disturbances is formulated as a generalized eigenvalue problem thanks to a convenient eigenmode decomposition. The ‘stiffness’ and ‘mass’ matrices are then straightforwardly obtained through a Galerkin projection. Finally, the SLEPc library using a shift-and-invert technique is employed to compute the generalized eigenpairs. Influence of the grid density, confinement due to the domain radius \(H\), position \(L^2\) of the outlet plane and entrance length \(L_1\) (see Figure 2) on the results is discussed in Appendix A.

The whole method was originally introduced by Sipp & Lebedev (2007) to study the stability of two-dimensional flows. It was subsequently adapted by Meliga et al. (2009b) to study the wake of fixed three-dimensional bodies, and by Assemat et al. (2012) to study the linear dynamics of two-dimensional freely falling bodies. Additional details and validations can be found in these references.

2.4. Base flow

The zeroth-order flow, also known as the base flow, is sought in the form of a steady axisymmetric flow associated with a steady vertical broadside motion of the disc. The corresponding state vector reads merely \(\mathcal{Q}_0 = [V_0(r), P_0(r), -x, 0, 0]\) (owing to the rotational symmetry of the body geometry, the last component of this state vector could contain an arbitrary constant angle \(\xi\) in the \(x\)-direction; we set it to zero for simplicity).

The nonlinear set of equations governing this base flow reads

\[
\nabla \cdot V_0 = 0 ,
(2.7)
\]

\[
(V_0 + x) \cdot \nabla V_0 = -\nabla P_0 + Re^{-1} \nabla^2 V_0 .
(2.8)
\]

It has to be supplemented by the no-slip condition \(V_0 = -x_b\) on \(\mathcal{S}\), the far-field condition \(V_0 = 0\) on \(\mathcal{S}_\infty \equiv \mathcal{S}_a \cup \mathcal{S}_b\) (see Figure 2), the symmetry condition \(V_0 = \partial_r V_{0r} = 0\) on the domain axis \(\mathcal{S}_a\), and the zero-traction condition \(-P_0 \mathbf{x} + \rho \nu \mathbf{x} \cdot \nabla V_0 = 0\) on \(\mathcal{S}_{out}\).

Figure 3 shows an example of the base flow seen in the laboratory reference frame in the case of an infinitely thin disc falling at a Reynolds number \(Re = 117\). As one could expect, the wake structure is exactly that of the flow past a fixed disc, once the uniform flow at infinity has been removed. The drag coefficient \(C_D \approx 1.20\) compares well
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with the value $C_D \approx 1.23$ determined experimentally by Roos & Willmarth (1971); the recirculation length, $l_R$, obtained by determining the point of the symmetry axis where $V_x = -1$, is $l_R \approx 2.2$, in good agreement with the value $l_R \approx 2.1$ obtained numerically for $Re = 116.9$ by Meliga et al. (2009a).

Although we retain the Reynolds number $Re$ based on $U_0$ as one of the control parameters, comparisons with experiments have to be carried out in terms of the Archimedes number $Ar$ defined above. The relation between $Re$ and $Ar$ is obtained from (2.3), which then reduces to a balance between the net weight of the disc and the drag force, namely

$$(M - \rho V_0)g = -\int \mathbf{T}_0 \cdot \mathbf{n} dS \equiv -D_0 \rho dU_0^2 x ,$$

which in non-dimensional form leads to

$$Ar^2 = \frac{3}{32} Re^2 C_D(Re) ,$$

where $C_D = 8 D_0 / \pi$ denotes the dimensionless drag coefficient. This correspondence between $Re$ and $Ar$ holds as long as the primary threshold at which the steady vertical body motion is destabilized is concerned. In contrast, it is no longer exact for subsequent transitions, since the corresponding base flow is not axisymmetric any more and involves a non-vertical body path. In these regimes, it is customary to introduce another Reynolds number, $Re_m$, based on the time-averaged settling/rise body velocity. However, the three-dimensionality of the base flow in the body wake results in higher values of the drag coefficient, so that the value of $Re_m$ corresponding to a given $Ar$ is lower than that predicted by (2.10) (see Figure 1 in Auguste et al. (2013)).

2.5. Global mode analysis

At order $\epsilon$, the state vector reads $\mathbf{q} = [\mathbf{v}(r,t), p(r,t), \mathbf{u}(t), \omega(t), \xi(t)]$ and the linearized perturbation equations take the form

$$(\nabla \cdot \mathbf{v}) = 0 ,$$

$$\partial_t \mathbf{v} + (\mathbf{V}_0 + x) \cdot \nabla \mathbf{v} + (\mathbf{v} - \mathbf{w}) \cdot \nabla \mathbf{V}_0 + \omega \times \mathbf{V}_0 = -\nabla p + Re^{-1} \nabla^2 \mathbf{v} ,$$

$$16 \frac{I^*}{1 + \frac{2}{3} \chi} \left( \frac{d\mathbf{u}}{dt} - \mathbf{x} \times \omega \right) - D_0 (\theta_x \mathbf{x} - \theta_y \mathbf{z}) = \int \mathbf{t} \cdot \mathbf{n} dS ,$$

$$I^* \frac{d\omega}{dt} = \int \mathbf{r} \times (\mathbf{t} \cdot \mathbf{n}) dS ,$$

$$\frac{d\xi}{dt} = \omega .$$
where \( t \) is the disturbance stress tensor and \( \theta_y \) and \( \theta_z \) are the disturbance pitch and yaw angles, respectively. In (2.13), the last term in the left-hand side represents the \( O(\epsilon) \)-contribution of the net body weight; its expression is obtained by making use of (2.9) and of the projection of the gravity vector in the body frame, namely \( g = -g_x 0 = -gx + \epsilon (\theta_y y - \theta_z z) \).

The system (2.11)-(2.15) must be supplemented by the no-slip condition \( v = w = u + \omega \times r \) on \( S \), the far-field condition \( v = 0 \) on \( S_\infty \), the zero-traction condition on \( S_{out} \), plus suitable conditions on the symmetry axis \( S_a \) to be specified later.

The solution of (2.11)-(2.13) is then sought in the form of normal modes, namely

\[
\mathbf{q} = \begin{pmatrix} \hat{u}^f(r, x) e^{im\phi} \\ \hat{\omega}^f(r, x) e^{im\phi} \\ \hat{\theta}^f(r, x) e^{im\phi} \end{pmatrix} e^{\lambda t} + c.c.,
\]

where c.c. denotes the complex conjugate, and \( \lambda = \lambda_r + i\lambda_i \) is the associated complex eigenvalue whose real and imaginary parts are the growth rate and frequency of the mode, respectively.

As mentioned above, the ‘fluid’ components \( \hat{u}^f \) of the eigenvector are expressed in the local cylindrical coordinate system \((r, \phi, x)\), using the azimuthal wavenumber \( m \) to remove the \( \phi \)-dependence, while the ‘solid’ component \( \hat{q}^b \) are expressed in the Cartesian system \((x, y, z)\).

Among the potentially nine components of \( \hat{q}^b = [\hat{u}, \hat{\omega}, \hat{\theta}] \), only a few have to be actually retained in the analysis. Their number depends upon the azimuthal wavenumber \( m \). To identify the relevant components, we may consider the symmetries of the hydrodynamical force and torque involved in the right-hand sides of (2.13) and (2.14), respectively.

- For \( m = 0 \) (axisymmetric modes), the hydrodynamical force and torque are held by the axial direction \( x \) and hence can only be coupled to the kinematic degrees of freedom corresponding to this direction. It is thus natural to take the corresponding solid components of the eigenmodes as \( \hat{q}^b = [\hat{u}_x, \hat{\omega}_x] \) (the roll angle might also be included but it can actually be dropped owing to the rotational invariance of the problem).
- For \( m = \pm 1 \) (helical modes), the hydrodynamical force and torque are held by the vectors \( y \pm iz \) (up to complex conjugates). Projections of (2.13)-(2.15) along \( y \) and \( z \) can thus be combined so as to obtain a single equation for each of (2.13)-(2.15) in the plane of the disc. This is achieved by introducing the so-called U(1)-coordinates (Jenny & Dusek 2004) in the form \( \hat{u}_\pm = \hat{u}_y \mp i\hat{u}_z, \hat{\omega}_\pm = \hat{\omega}_y \mp i\hat{\omega}_z \) and \( \hat{\theta}_\pm = \hat{\theta}_y \pm i\hat{\theta}_z \). The solid components associated with the eigenmodes for \( m = \pm 1 \) can thus be reduced to three complex numbers, namely \( \hat{q}^b = [\hat{u}_m, \hat{\omega}_m, \hat{\theta}_m] \).
- For \( |m| \geq 2 \), the overall force and torque induced by the fluid component of the eigenmode vanish upon integration with respect to \( \phi \) over \([0, 2\pi]\). Thus, there is no coupling between the fluid and the body for these modes, and the solid component \( \hat{q}^b \) can thus be simply dropped from the problem, which becomes identical to the one governing the stability of the flow past a fixed body.

The problem (2.11)-(2.15) with appropriate boundary conditions can then be recast as a generalized eigenproblem in the form

\[
\mathcal{A}_m \mathbf{q} = \lambda \mathcal{B}_m \mathbf{q},
\]
where
\[ \mathcal{A}_m = \left( \mathcal{A}_m^I(Re) \mathcal{A}_m^b(Re, \mathcal{C}) \right), \quad \mathcal{B}_m = \begin{pmatrix} I & 0 \\ 0 & \mathcal{B}_m^b(\mathcal{I}^*) \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} \hat{q}_1^I \\ \hat{q}_1^b \end{pmatrix}. \]

The matrices \( \mathcal{A}^I \) and \( \mathcal{B}^I \) represent the linear operators acting only on the fluid variables \( \hat{q}^I = [\hat{v}, \hat{p}] \). Similarly, \( \mathcal{A}^b \) and \( \mathcal{B}^b \) are the operators of the linearized rigid-body equations which only act on the disc kinematic variables \( \hat{q}^b = [\hat{u} \pm \hat{\omega} \pm \hat{\theta} \pm] \). Terms \( \mathcal{C} \) and \( \mathcal{F} \) ensure the coupling between the body and fluid. The former expresses the action of the disc motion on the fluid flow (through the no-slip boundary condition on \( \mathcal{C} \) and the presence of the body velocity and rotation rate in the Navier-Stokes equation), while the latter expresses to role of the fluid on the body motion through the hydrodynamic force and torque in (2.13) and (2.14), respectively. All submatrices are detailed in Appendix B, as well as the symmetry conditions on the axis \( \mathcal{I}^* \) to be retained for each value of \( m \). As shown in Appendix (B), the case \( m = 0 \) yields only stable modes whose properties and spatial structure are discussed in Appendix C. For \( |m| = 2 \) instabilities were found to exist only beyond \( Re \approx 180 \), which is much higher than the values of the primary thresholds corresponding to \( |m| = 1 \). The behaviour of these \( |m| = 2 \) modes is also briefly discussed in Appendix C. For the reasons just stated, we will restrict the discussion in the rest of the main text to the case \( |m| = 1 \), or rather to \( m = +1 \) by taking advantage of the symmetries \( (v_r, v_p, v_z, p, u_x, u_y, \omega_z, \theta_z, m) \to (v_r, -v_p, v_z, p, u_x, u_y, \omega_z, \theta_z, -m) \).

Note that modes with \( m = +1 \) and \( \lambda_i > 0 \) correspond to a right-handed helix, while those with \( m = -1 \) and \( \lambda_i > 0 \) correspond to a left-handed helix. In the framework of linear theory, both kinds of helices are admissible solutions, as well as any linear superposition of them. In particular, the superposition of right and left-handed helices of equal amplitudes yields a planar zigzagging path.

3. The infinitely thin disc

3.1. Parametric study

We start discussing the results of the stability analysis by examining the limit case of an infinitely thin disc. Strictly speaking, the disc we consider corresponds to \( \chi = 10^4 \) because the FreeFem++ solver requires the body to have a nonzero thickness. At such very large aspect ratios the precise value of \( \chi \) has no influence, as shown by Meliga, Chomaz & Sipp (2009a) and Fabre et al. (2012) who employed the same software: considering discs with \( \chi = 10^3 \) and \( \chi = 10^4 \), respectively, they recovered with an excellent accuracy the thresholds computed by Natarajan & Acrivos (1993) using a strictly infinitely thin disc. We shall also show below that present results fully corroborate numerical data corresponding to the limit \( \chi \to \infty \).

The first important finding of the LSA is that at least four modes exist whatever \( \mathcal{I}^* \). One of them is stationary while the other three are oscillating global modes. Hereinafter, these modes are identified as \( F_1, F_2, S_1 \) and \( S_2 \), with \( S \) referring to ‘solid’ and \( F \) to ‘fluid’ for reasons that will become evident later. Figure 4 illustrates the situation in the case of a disc with \( \mathcal{I}^* \approx 4 \times 10^{-3} \) by displaying the variations of the growth rate \( \lambda_i \) and Strouhal number \( St = \lambda_i / 2\pi \) with the Reynolds number. The \( S_2 \) mode is stationary (\( \lambda_i = 0 \)) while \( F_1, F_2 \) and \( S_1 \) are oscillating (thus corresponding to pairs of complex conjugates eigenvalues with \( \lambda_i \neq 0 \)).

In the case of a disc held fixed in a uniform stream, the growth rate \( \lambda_i \) changes from negative to positive values at a critical Reynolds number \( Re_c \) and keeps positive values
Figure 4: Critical eigenvalues for $I^* = 4 \cdot 10^{-3}$ and $Re \leq 220$. The growth rate (resp. frequency) of modes $F_2$, $S_1$, and $S_2$ is depicted with full (resp. dashed) lines. The plotted frequency is actually a Strouhal number defined as $St = \lambda_i / 2\pi$.

for larger $Re$. Figure 4 shows that more subtle scenarios can exist for a freely moving disc. Indeed the $S_1$ mode first becomes unstable for $Re \approx 104.4$, restabilizes at $Re \approx 136$ and destabilizes again at $Re \approx 167.7$. This ‘destabilization-restabilization’ process, which is not observed in other ranges of $I^*$, was also noticed in the LSA of two-dimensional freely moving plates and rods (Assemat et al. 2012). We systematically detected all thresholds over a fairly large range of the control parameters $I^*$ and $Re$. The four branches where a change of path occurs in a LSA perspective are gathered in Figure 5.

The gray-shaded area in that figure is the zone where a rectilinear vertical path is stable with respect to infinitesimally small disturbances. The curve corresponding to the $S_1$ branch shows that the destabilization-restabilization process mentioned earlier is actually confined within a narrow interval, typically $I^* \in [3.2 \cdot 10^{-3}, 5 \cdot 10^{-3}]$. The picture provided by Figure 5 is exhaustive since all curves end with horizontal asymptotes corresponding to the limits $I^* \to \infty$ and $I^* \to 0$, respectively. This stability diagram allows us to identify three subregions corresponding respectively to low, moderate and high inertia ratios, which we approximately define as $I^* \leq 5 \cdot 10^{-3}$, $5 \cdot 10^{-3} \leq I^* \leq 1.0$ and $I^* \geq 1.0$, respectively. The marginal curve corresponding to the minimum critical Reynolds number for each value of $I^*$ is made of mode $F_1$ for $I^* \leq I^*_c \approx 3.2 \cdot 10^{-3}$ and of mode $S_1$ for larger $I^*$. This mode switching has a spectacular consequence, as it is associated with a frequency jump from $St \approx 0.1$ to $St \approx 0.25$ as $I^*$ crosses the critical value $I^*_c$; at large inertia ratios, the frequency of the $S_1$ mode is found to behave as $I^{*1/2}$ (see Figure 14(b)).

A surprising feature is the existence of a small stable subregion around $Re \approx 95, I^* \approx 0.09$. This subregion correspond to a ‘loop’ of the $F_1$ branch which turns back twice through two saddle nodes; accordingly, the lower part of the critical curve that joins those two points corresponds to a restabilization of the $F_1$ mode. This feature is qualitatively similar to the destabilization/restabilization event occurring along the $S_1$ branch, as illustrated in Figure 4. According to Figure 5(b), the frequency of the mode experiences a sharp variation in the range corresponding to this ‘loop’.

Finally, LSA predicts that the stationary $S_2$ mode becomes unstable at $Re = 141.5$, i.e. $Ar \approx 45.57$ (see Figure 4), regardless of the disc inertia. This conclusion is in line
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Figure 5: Neutral curves for an infinitely thin disc as a function of dimensionless inertia $I^*$. (a): Reynolds number, (b) Strouhal number. The continuous part of the $S_1$ curve corresponds to the range within which a destabilization-destabilization process takes place; the vertical arrow in Figure 5(b) indicates the frequency jump associated with the switching from mode $F_1$ to mode $S_1$ along the critical curve.

with the weakly nonlinear predictions of Fabre et al. (2012) who determined that discs switch from a strictly vertical path to a steady oblique (SO) path (which results from the addition of a stationary mode to the base flow) exactly for this value of $Re$. That the $S_2$ branch is independent of $I^*$ could have been inferred from the fact that inertia is not involved in (2.11)-(2.15) any more when $\lambda = 0$, as is the case at criticality with a stationary bifurcation.

3.2. Comparison with available studies

The complex behaviours displayed in Figure 5 are far from being anecdotal. In particular, their comparison with the DNS data reported by Auguste et al. (2013) sheds light on a number of so far unexplained results. To evidence this, we made use of (2.10) to express
Figure 6: Comparison of thresholds provided by LSA (thin lines) with the regime map obtained through DNS by Auguste et al. (2013). Seven well-characterized regimes may be identified according to DNS data: the large area on the left (gray online) corresponds to the steady vertical fall; then increasing $I^*$ along the right vertical axis of the figure one successively encounters a planar zigzagging (or fluttering) path (green online) hereinafter termed as the ZZ regime, a helical zigzagging path termed ‘hula-loop’ by Auguste et al. (cyan online), a chaotic intermittent fluttering/tumbling regime (purple online), a tumbling regime (orange online) and a helical tumbling regime (brown online). The thick (red online) curve encloses the zone where small-amplitude ‘A-regimes’ are observed; the small triangle at its bottom left corner (blue online) corresponds to a small-amplitude zigzagging path (hereinafter termed as the ZZ$_2$ regime) with a frequency typically three times smaller than that of the main fluttering path.

The thresholds in terms of $Ar$ rather than $Re$ and switched the vertical and horizontal axes of Figure 5(a) so as to obtain a direct comparison with the phase diagram displayed in Figure 2 of Auguste et al. This procedure resulted in Figure 6 which reveals that the marginal curve provided by the LSA matches the loss of the steady vertical path predicted by the DNS quite well, especially in the range $9 \cdot 10^{-3} \leq I^* \leq 3 \cdot 10^{-1}$. For lighter discs, the picture is complicated by the existence of small-amplitude regimes (called ‘A-regimes’ by Ern et al. (2012)). The lower bound of the range of existence of the fluttering regime also departs from the neutral curve predicted by LSA, owing to the subcritical nature of the corresponding transition (Auguste et al. 2013; Chrust et al. 2013). Nevertheless, the thick line corresponding to the transition from steady vertical fall to the small-amplitude regimes is in perfect agreement with the prediction of LSA.

A noticeable discrepancy is also observed for $I^* > 3.5 \cdot 10^{-1}$: while DNS indicates that two successive transitions (vertical/fluttering and fluttering/tumbling) take place within the narrow range $15 \lesssim Ar \lesssim 17$, LSA still predicts a stable vertical path in that range. Here again, the subcritical nature of these transitions was numerically attested.

Note that DNS confirms the existence of a stable vertical fall regime in a small region around $(Ar \approx 36.0, I^* \approx 0.09)$, which coincides with the stable subregion embedded within the aforementioned ‘loop’ formed by the $F_1$ branch. In this range, the vertical
path coexists with the large-amplitude tumbling motion but is only reached in DNS when using initial conditions very close to the vertical fall. Owing to this sensitivity to initial conditions, this isolated vertical path regime would certainly be extremely difficult to observe in experiments.

Apart from this peculiar feature, the higher branches predicted by the LSA do not seem to be relevant for determining the boundaries separating the various falling regimes. This is not surprising, since the linearity of this approach and the fact that it makes use of an axisymmetric base flow make its relevance for describing the transition between large-amplitude regimes highly questionable.

In a more qualitative manner, we can compare the LSA and DNS predictions for the structure of the axial vorticity field in the wake of a falling disc. This comparison is reported in Figure 7 for points P\(_1\) and P\(_4\) located along the marginal curve (see Figure 5). In both cases, the DNS predicts a supercritical transition to a fluttering (planar zigzag) path, with positive and negative vorticity alternately shed from each half of the back face of the disc. To achieve this comparison, we first performed DNS for these two sets of \((Ar, \text{I}^*)\) with the in-house JADIM code (Mougin & Magnaudet 2002a; Auguste et al. 2013). Then we superimposed the \(m = 1\) and \(m = -1\) helical modes predicted by the LSA onto the corresponding base flow, using an arbitrary but small amplitude for the former. The two types of approach yield the right and left snapshots in Figure 7, respectively. The resemblance of the two families of vortical structures is striking. These results also underline the two types of fluttering regimes mentioned in the caption of Figure 6: the wake of the disc corresponding to \(\text{I}^* = 1.5 \cdot 10^{-3}\) exhibits elongated vortices characteristic of the ZZ\(_1\) regime while that of the disc corresponding to \(\text{I}^* = 1.6 \cdot 10^{-1}\) reveals much shorter structures typical of the ZZ regime. More precisely the streamwise vortices are approximately three times longer in the ZZ\(_2\) regime, so that the Strouhal number associated with the shedding process is almost three times smaller than in the ZZ regime. That the LSA approach not only predicts the disc’s oscillations but is also able to identify the two types of fluttering regimes reinforces its relevance in view of the prediction of the characteristics of the first non-straight paths of falling bodies.

Table 1 displays for some values of \(\text{I}^*\) the values of \(Ar_c\) and \(St\) predicted by the present LSA and compares them with DNS data from Auguste et al. (2013) and Chrust et al. (2013) and with the quasi-static theory (QST) developed by Fabre et al. (2011). A brief account of the assumptions underlying this theory and of its main results is provided in Appendix D. Table 1 shows that the values of the Strouhal number \(St\) predicted by the LSA agree well with those reported in available DNS. In the small \(\text{I}^*\)-range, the two widely different frequencies reported in these studies and respectively associated with the ZZ and ZZ\(_1\) regimes are faithfully recovered. Figure 5(b) reveals that these two frequencies are associated with two different unstable branches of the eigensolutions of (2.17), the ZZ mode being related to either \(F_2\) or \(S_1\) while the ZZ\(_2\) mode is connected to \(F_1\). The situation is more complex for \(\text{I}^* = 4 \cdot 10^{-3}\) where DNS predicts both a subcritical ZZ regime that sets in at \(Ar = 33\) and is recovered by the LSA with a threshold located at \(Ar = 35.8\), and a supercritical ZZ\(_2\) regime that sets in at \(Ar = 34.5\). The latter is not directly predicted by the LSA; however the Strouhal number corresponding to this mode in the DNS is close to the one characterizing the \(F_1\) mode which becomes unstable at a somewhat larger value of \(Ar\). We suspect that this regime results from nonlinear interactions between modes \(S_1\) and \(F_1\) as it occurs for a value of \(\text{I}^*\) very close to a point of co-dimension 2 corresponding to the intersection of the aforementioned two branches. For moderate \(\text{I}^*\), LSA predictions and DNS results match well regarding both the pri-
Figure 7: Qualitative comparison of iso-surfaces of axial vorticity obtained by: superposing the modes $m = \pm 1$ predicted by the LSA onto the base flow (left), and through DNS close to the fluttering threshold (right); top: $I^* \approx 1.5 \cdot 10^{-3}$, bottom: $I^* \approx 1.6 \cdot 10^{-1}$.

mary threshold $Ar_c$ and the associated Strouhal number $St$. In contrast, the QST only captures properly the latter. This is in line with the findings of Assemat et al. (2012) who observed that QST correctly assesses the frequency down to moderate inertia ratios, whereas its predictions for the fluttering threshold are only reliable for large inertia ratios. That QST predictions for the fluttering threshold improve when $I^*$ increases is obvious in Table 1 (compare the predictions corresponding to $I^* = 1.6 \cdot 10^{-1}$ and $I^* = 5 \cdot 10^1$).

Note that for $I^* = 1.6 \cdot 10^{-1}$, DNS predicts existence of a supercritical large-amplitude fluttering regime, thus closer to the $ZZ$ type than to the $ZZ_2$ type despite a Strouhal number about 0.13. As shown in Figure 5(b), this is a range of $I^*$ within which the $S_1$ and $F_1$ modes exhibit quite similar frequencies. This situation underlines the obvious fact that when several modes have quite similar frequencies, the mere knowledge of $St$ provided by the LSA is not sufficient to predict the regime that is observed after nonlinear saturation. A weakly nonlinear analysis might help clarify this point. We shall see latter that useful information can also be obtained by examining the spatial structure of the unstable modes.

Results corresponding to $I^* = 5 \cdot 10^1$ show that the QST predictions compare well with those of the fully-coupled LSA for large inertia ratios, thus validating the uncoupling
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<table>
<thead>
<tr>
<th>$I^*$</th>
<th>$Re_c$</th>
<th>$St$</th>
<th>[Stability branch]</th>
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Table 1: Comparison of thresholds and frequencies at representative values of the inertia ratio.

of the body and fluid time scales on which QST is based (see Appendix D). These results show that crossing the $S_1$ branch results in a slow ($St \approx 0.008$) unsteady motion of the disc with a quasi-steady wake, a situation which will be illustrated in the next paragraph. Finally Table 1 shows that, for large $I^*$, the LSA recovers the existence of ‘wake’ (or ‘fluid’) modes. These are global modes which exist, with the same thresholds and frequencies, even if the disc is held fixed. Indeed, the first Hopf bifurcation for the flow past a fixed infinitely thin disc takes place at $Re_c \approx 125.3$ (Natarajan & Acrivos 1993; Meliga et al. 2009b) and oscillates at $St \approx 0.12$. Here this mode is found to be the asymptotic limit reached by the $F_1$ branch for large $I^*$. Also, we find that the asymptotic limit of the $F_2$ branch corresponds to a second oscillating mode. We performed a specific LSA of the flow past a fixed disc (with an independent stability code) and also observed this mode which sets in through a Hopf bifurcation at $Re_c \approx 272.1$ with $St \approx 0.22$, i.e. approximately a frequency twice that of the first ‘fluid’ mode. To our knowledge, this second Hopf mode has not been previously reported. Its physical relevance is of course questionable, since the base flow is no longer axisymmetric at such values of $Re$. This mode may be seen as the counterpart for the disc of the second von Kármán mode predicted by LSA in the wake of two-dimensional fixed bodies (Assemat et al. 2012). That modes $F_1$ and $F_2$ may be identified with the two global modes observed past a fixed disc in the limit $I^* \to \infty$ justifies the terminology of ‘fluid’ modes, while modes $S_1$ and
To summarize, comparison of LSA predictions with available DNS results reveals the relevance of the former approach which is found to properly recover the thresholds, frequency and spatial structure of most unstable modes detected in the DNS near the first unstable threshold. A noticeable gain of the LSA is the clear view it provides on the various unstable branches of the eigensolutions and on their asymptotic behaviours in the limit of small and large inertia ratios as well as on their possible crossings at specific values of $I^*$. The main issue we identified is the difficulty of a direct distinction between the ZZ and ZZ$_2$ regimes which strongly differ by their amplitude in the saturated state but may have similar frequencies in some range of $I^*$. We shall come back to this point later by examining the spatial structure of the oscillating modes.

3.3. Global modes structure: segregation between fluid and body influences

We now investigate the structure of the unstable modes by selecting several points along the various branches of the stability diagram in the ($I^*, Re$) plane. Points $P_1$ to $P_6$ marked in Figure 5 are chosen so as to gain some more insight into the nature of the observed disc motion and characteristics of the fluid-body coupling. Figure 8 displays at each point $P_i$ the real (left column) and the imaginary (right column) parts of the global mode structure, normalized by the inclination angle $\theta_z = \theta_z \pm \theta_y$. Given this definition of the inclination, the left column displays the modes in a state corresponding to the maximum of $\theta_z$, whereas the right column corresponds to a state with $\theta_z = 0$ (and to a maximum $\theta_y$ in the case the path is tridimensional). Note also that when $\theta_z = 0$, the $z$-component of the rotation rate is maximum. The great advantage of this representation is that it provides a way to observe a given unsteady mode at two different instants of time; the case of a steady mode is slightly more subtle and will be detailed in section 4.

Both the real and the imaginary parts of the $F_1$ global mode at $P_1$ exhibit an alternation of positive and negative disturbances (enlightened by the streamline pattern) which is a clear footprint of wake oscillations. This wake structure very much resembles that behind a fixed disc (e.g. the ‘fluid’ mode at $P_6$ displayed in the last row), a feature worthy of interest since the point $P_1$ corresponds to a low value of $I^*$, i.e. to a situation where the disc is rather expected to be very sensitive to flow disturbances. Although surprising at first glance, this characteristics is in line with DNS observations where the first departure from the steady vertical fall in this range of $I^*$ ($I^* \lesssim 3 \cdot 10^{-3}$) corresponds to the low-amplitude (or quasi-vertical) ZZ$_2$ regime (Auguste et al. 2013; Chrust et al. 2013), a regime that can hardly be properly characterized in experiments owing to residual disturbances in the fluid (Fernandes et al. 2007; Ern et al. 2012).

Although it corresponds to the same value of $I^*$, the $F_2$ mode at $P_2$ reveals an utterly different behaviour. This mode has a clear oscillating nature, as evidenced by the axial vorticity distribution in the near wake. However the axial velocity distribution dramatically depends on which instant is chosen to observe it: while it takes the form of an elongated strip with a constant sign extending far downstream when $\theta_z = 1$, it reduces to a few rolls of alternating sign confined to the near-wake when $\theta_z = 0$. Hence, unlike what we observed at $P_1$, a dramatic reorganization of the structure of the eigenmode within the wake takes place during the period of time separating the two snapshots. We interpret this reorganization as the footprint of the mutual coupling between the disc’s degrees of freedom and its wake, a feature that appears to be much stronger on the $F_2$ mode at $P_2$ than in the previous case where the wake dynamics barely affects the motion of the disc. This corroborates the DNS results reported in Figure 6. Indeed, for
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Figure 8: Global modes (m=1) along the various branches of the stability diagram displayed in Figure 5. Each row in the figure corresponds to a single point $P_i$, from $i = 1$ at the top to $i = 6$ at the bottom. The upper (resp. lower) half in each snapshot displays the axial velocity (resp. axial vorticity); streamlines are also drawn in the upper half. The left (resp. right) column show the real (resp. imaginary) part of the modes that have been normalized by the complex inclination $\theta_\star$. The six investigated points have the following $(I^*, Re)$ coordinates: $P_1(2 \cdot 10^{-3}, 113)$, $P_2(2 \cdot 10^{-3}, 161.2)$, $P_3(5 \cdot 10^{-3}, 94)$, $P_4(1.6 \cdot 10^{-1}, 33)$, $P_5(50, 59.5)$, $P_6(50, 273)$.

$I^* \approx 1.5 \cdot 10^{-3}$, the DNS map predicts a transition from the $A$-regimes (enclosed within the thick line) to the large-amplitude $ZZ$-fluttering regime for $Ar \approx 55$, a value that compares well with the threshold value $Ar = 53$ predicted by the LSA at $P_2$ (the LSA prediction for the Strouhal number, $St \sim 0.29$, also agrees well with the DNS result). The above interpretation of the differences revealed through the real and imaginary parts of the modes is also supported by what can be observed at point $P_3$ and even more at $P_4$. 
the latter corresponding to the ‘shortened’ oscillating wake displayed in the second row of Figure 7. Given that the transition at $P_4$ is known from DNS to yield a supercritical large-amplitude $ZZ$ regime despite a frequency ($St \approx 0.13$) closer to that of mode $F_1$ at $P_1$ than to that of mode $F_2$ at $P_2$, we conclude that the similarities in the spatial structure of the associated global mode at points $P_2$, $P_3$ and $P_4$ allow us to anticipate that all three of them are characterized by a strong fluid-body coupling. However, based on the qualitative degree of structural changes between the real and imaginary parts on the one hand and on the maximum of velocity and vorticity isovalues on the other hand, this coupling is likely to be stronger at $P_2$ and $P_3$ than it is at $P_4$.

More precisely, our statement is that all regimes in which the path of the disc exhibits large-amplitude deviations from the vertical result from strong interactions between the body and its wake and share the linear signature unambiguously observed at point $P_4$. This common signature is such that the associated global mode successively exhibits disturbances of ‘sign alternating type’ (SAT) and of ‘sign preserving type’ (SPT). The former (Figure 8(h)) involve rolls corresponding to clockwise and anti-clockwise fluid motions which are intense only in the near wake, say up to $x \approx 3$, and then decrease downstream. In contrast the latter (Figure 8(g)) take the form of an elongated strip of constant sign located along the wake axis, with only weak rolls aligned along the outer edge of this central region. The resemblance between the SPT wake structure and that of the wake behind a fixed disc just beyond the first bifurcation, which is known to yield nonzero stationary lift and torque, suggests that the effect of the SPT disturbances is to deviate the wake from its original orientation. When the disc moves freely, this deviation results in a drift between its geometrical axis and its translational velocity. Therefore a periodic motion of the disc can be understood from the succession of SAT and SPT disturbances, the former being responsible for fluid oscillations in the wake at each disc inclination, the latter modifying the disc inclination without much fluid oscillations.

The spatial structure of the global mode at $P_5$ on $S_1$ (Figure 8(i) – (j)) is purposely represented over a very large domain downstream of the disc. This allows us to see the subtle switch from SPT to SAT structures that would be missed, had the mode structure been displayed over the same domain as in the previous figures. The snapshot 8(j) shows that the wavelength of this SAT structure is very large. It then results in very slow oscillations, in agreement with the low frequency predicted by the QST. Moreover, the amplitude of the fluid velocity disturbance associated with this mode is weak, which suggests that it acts more on the disc than on the fluid. On the basis of the above linear criteria, this slow change from SPT to SAT structures implies that the coupling is very weak, although nonzero, the disc’s influence manifesting itself only over a ‘long’ time scale. This is consistent with the separation of time scales at the root of the QST which qualifies this mode as ‘aerodynamic’ or ‘solid’, as opposed to the ‘fluid’ modes displayed in snapshots 8(k) and (l). In contrast with the previous case, both the real and imaginary parts of velocity and vorticity disturbances reach very large amplitudes in the wake at $P_6$. Therefore, the corresponding mode virtually acts only on the fluid, thus belonging to the ‘fluid’ category. The $\theta_z$-independent behaviour of the SAT rolls revealed by these two snapshots confirms that there is almost no coupling between the disc and fluid motions for such large inertia ratios.

Last, we may notice that all these results indicate that a strong fluid-body coupling is only observed with discs of low or moderate relative inertia, modes $P_2$–$P_4$ belonging to the range of moderate $I^*$ while $P_5$ corresponds to a large value of $I^*$. The peculiar case of $P_1$ which belongs to the low-$I^*$ range but displays a weak coupling behaviour underlines the complexity of the entire problem.
Figure 9: Neutral curves for a disc with $\chi = 10$. Same conventions as in Figure 5. The dashed vertical line at $I^* = 4.908 \cdot 10^{-3}$ in (a) corresponds to discs with the same density as the fluid ($\bar{\rho} = 1$) and thus separates ‘light’ rising discs from ‘heavy’ falling discs.

4. A thin disc with $\chi = 10$

We now consider the case of a disc of finite thickness such that $\chi = 10$. This specific aspect ratio has been used in several previous experimental and computational studies, e.g. Fernandes et al. (2007); Auguste et al. (2013) which focused on density ratios slightly lower than unity, i.e. on inertia ratios in the range $4 \cdot 10^{-3} \leq I^* \leq 5 \cdot 10^{-3}$. Selecting the same geometry will allow direct comparisons with these data and facilitate the interpretation of the stability branches.

Figure 9 gathers the four unstable branches of the eigensolutions of (2.17). They are identified using the same terminology as for the infinitely thin disc. Comparing the neutral curves displayed in this figure with those of Figure 5 for an infinitely thin disc reveals striking differences. Not only is the critical Reynolds number shifted towards higher values whatever $I^*$ (see the general stability diagram in Figure 14), but also the nature of the mode involved in the first destabilization differs when $I^*$ is small. This is noteworthy
because experiments have long assumed that the disc aspect ratio does not play any role in the dynamics of the system, provided it is ‘sufficiently’ large, which led to the building of regime maps gathering results obtained with discs of widely different aspect ratios, most of which in the range $10 \leq \chi \leq 10^2$ (Willmarth et al. 1964; Field et al. 1997). According to LSA predictions, the first bifurcation leading to a non-straight path for a thin disc with $\chi = 10$ is stationary for $I^* \leq 2 \cdot 10^{-2}$ (which corresponds to the $S_2$ branch in Figure 9), leading to a steady oblique path. This is in stark contrast with what we observed in Figure 5 where the first bifurcation in this range of $I^*$ is of Hopf type and thus leads to an oscillatory path. That the nature of the first non-vertical path of low-inertia discs crucially depends on their aspect ratio, even when it may be thought to be ‘large’, is in full agreement with the DNS results of Auguste et al. (2013) who provided a detailed comparison of the transition sequence for an infinitely thin disc and a disc with $\chi = 10$, both with $I^* \simeq 4 \cdot 10^{-3}$, and indeed found that the first non-vertical path of the latter is steady oblique while that of the former is time-dependent.

The crossing of modes $S_2$ and $F_1$ and of modes $F_1$ and $S_1$ along the critical marginal curve results in two frequency jumps (gray arrows in Figure 9(b)). Again, a destabilization-restabilization subregion is found to exist on the $S1$ branch in the range $1.6 \cdot 10^{-1} \leq I^* \leq 4.5 \cdot 10^{-1}$. For larger $I^*$, the stability diagram is qualitatively similar to that of an infinitely thin disc, with branches $S_1$, $F_1$, $S_2$ and $F_2$ successively crossed as $Re$ increases. In the limit $I^* \to \infty$, the thresholds of branches $F_1$, $F_2$ and $S_1$ are found to be $Re \simeq 138.6$, $Re \simeq 274$ and $Re \simeq 78.6$, respectively. The independent LSA study we performed for the case of a fixed disc with $\chi = 10$ confirmed the previous two values (which justifies that modes associated with branches $F_1$ and $F_2$ be termed ‘fluid’), while the latter was recovered using the QST approach which again predicts existence of a low-frequency fluting mode in the large-$I^*$ limit.

Let us now investigate the structural features of these modes at points $P_1$ of the various branches of the stability diagram in Figure 9(a). The axial velocity and vorticity of these modes are displayed in Figure 10 using the same normalization as in Figure 8. The first row corresponding to point $P_1$ shows that the structure of the corresponding primary mode requires a non-zero $\theta_0$, since its imaginary part is uniformly null in our unit-inclination normalization. This global mode is thus tied to the tilt of the body and any disturbance in the fluid is merely a consequence of the disc being inclined. Since the SPT- and SAT-type disturbances do not exist in this case, our previous criterion to assess the fluid-body coupling does not properly apply here, because there is no actual instant of time at which the body is uninclined. Nevertheless, it is clear that the stationary mode at $P_1$ corresponds to a one-way coupling, the wake being enslaved to the body.

Figure 11 shows how the spatial structure of the axial vorticity of this global mode compares with that provided by the DNS. The stationary mode consists of two time-independent counterrotating vortices, leading to a permanent drift of the disc. This result in the so-called steady oblique (SO) path whose occurrence and characteristics were predicted by Fabre et al. (2012) through a weakly nonlinear analysis. While the axial velocity strip well visible in Figure 10(a) for $x \gtrsim 3$ results from the non-zero incidence angle of the disc, the existence of the standing eddy in the near wake ($x \lesssim 2$) helps explain how the zero-torque condition is satisfied along this steady path. Owing to the antisymmetry of modes $m = \pm 1$, the negative axial velocity disturbance seen in the upper half of Figure 10(a) for $x \lesssim 2$ is positive in the lower half-plane, so that the negative velocities in the primary toroidal eddy are strengthened above the symmetry axis and weakened below it, yielding a larger drag on the disc in the upper-half plane and thus a positive torque tendency to increase the disc’s inclination. This effect is balanced by
the so-called ‘restoring’ added-mass torque which tends to realign the disc’s velocity and symmetry axis, yielding an inclined path with a zero net torque.

Still for the same value of $I^*$, the structure of the global mode at $P_2$ shows that crossing the $F_2$ branch in this range of inertia ratios would lead to path oscillations caused by a rather strong mutual coupling between the body and fluid since the axial velocity disturbance switches from a SPT structure in snapshot (c) of Figure 10 to a SAT structure in snapshot (d). This suggests rather large saturated amplitudes, in line with the DNS results which predict a ZZ flutter after a succession of A-regimes starting with the
Figure 11: Axial vorticity iso-surfaces in the first bifurcated state of the flow past a disc with \( \chi = 10 \) for \( I^* = 5 \cdot 10^{-3} \) and \( Re = 144.2 \). LSA (left) vs. DNS by Auguste et al. (2013) (right).

above steady oblique path. The correspondence with DNS predictions extends to the threshold and frequency: LSA predicts the \( F_2 \) mode to become unstable with \( St = 0.22 \) at \( Re \approx 231.4 \) (corresponding to \( Ar \approx 66.8 \)), which compares fairly well with the DNS predictions \( St = 0.205, Ar \approx 63.5 \) (Auguste et al. 2013) and experimental observations \( St = 0.24, Ar = 70 \pm 3 \) (Fernandes et al. 2007). The slightly lower threshold detected in the DNS is due to the subcritical nature of the corresponding bifurcation which was evidenced by Auguste et al. (2013) and Chrust et al. (2013).

Again, the marked differences between snapshots (e) and (f) indicate that the unstable \( F_1 \) mode at \( P_5 \) is characterized by a strong fluid-body coupling. Note that the corresponding oscillating regime has a significantly lower frequency compared to that at \( P_2 \), as underlined by the larger streamwise spacing between the SAT disturbances (compare snapshots (d) and (f)). The latter trend is also present at \( P_3 \) and even more at \( P_5 \) which both belong to the \( S_1 \) branch along which \( St \) evolves as \( I^{*-1/2} \) at large \( I^* \). Therefore, on this branch, the streamwise extent of each roll in the axial velocity disturbance increases with \( I^* \), making the number of rolls present between the disc and a given downstream location decrease. At large enough \( I^* \), this trend results in the presence of a single roll in the fraction of the wake displayed in the snapshots, as may be observed in snapshots (i) and (j), although the size of the domain in the streamwise direction is about six times larger in these two subfigures than in the previous ones. Nevertheless, comparing snapshots (i) and (j) indicates that the shape of this single roll changes, depending on whether the real or the imaginary part is considered. Hence the disturbance is still time-dependent, in contrast with that at \( P_1 \). Still in contrast with the behaviour at \( P_5 \), the real and imaginary parts of the axial velocity disturbance do not change sign in the near wake at \( P_5 \) (also at \( P_5 \) in Figure 8).

The \( F_1 \) mode at \( P_6 \) (snapshots (k) and (l)) is qualitatively similar to its counterpart in Figure 8: it is a pure fluid mode with virtually no influence on the disc motion (see the magnitude of the normalized velocity and vorticity disturbances), whose characteristics match those of the first oscillating global mode past the disc held fixed. Last, although \( P_3 \) and \( P_6 \) both belong to the same \( F_2 \) branch, modes found along this branch deserve to be termed ‘fluid’ only in the large-\( I^* \) limit, since we found that the unstable mode at \( P_2 \) bears the mark of a significant mutual fluid-body coupling. Note that, on the basis of the differences in the geometry of the axial velocity and vorticity isocontours and in the
corresponding isovalues, this coupling at $P_2$ is expected to be weaker than that at $P_3$ and $P_4$, but stronger than that at $P_5$. This variation in the strength of the coupling with $I^*$ in similar to that noticed in the case of an infinitely thin disc: the moderate-$I^*$ range is that where the coupling appears to be the strongest, followed by the low-$I^*$ range and at last by the range corresponding to large inertia ratios. This suggests that, in the spirit of the QST, a low-$I^*$ theoretical model might be derived on the ground of a weak-coupling hypothesis.

5. A thick disc with $\chi = 3$

We finally consider a thick disc or thin cylinder with an aspect ratio $\chi = 3$. This particular geometry has been used in several experimental and computational studies with an inertia ratio $I^* = 1.6 \cdot 10^{-2}$ (corresponding to a body-to-fluid density ratio of 0.99). Although specific to this value of $I^*$, the corresponding findings provide a basis of comparison for the LSA results.

The stability diagram gathering the four neutral curves is displayed in Figure 12. As with the previous two geometries, three of these curves are associated with a Hopf bifurcation while the fourth corresponds to a steady bifurcation. The critical curve is found to consist of the $F_1$ branch for $I^* \lesssim 0.28$, then the $S_2$ branch until $I^* \approx 1$, and finally the $S_1$ branch for larger $I^*$. The frequency associated with this curve experiences two jumps encountered at those values of $I^*$ corresponding to the crossing of the branches $S_2$ and $F_1$ on the one hand, $S_2$ and $S_1$ on the other hand. In contrast with the case of thin discs, the stability diagram does no longer display any destabilization-re-stabilization region. In the whole range of inertia ratios, the $S_1$ branch is associated with low Strouhal numbers such that $St \leq 0.05$, while such a range was only encountered when $I^* > 1$ for thin discs. Last but not least, in the limit $I^* \to \infty$, the thresholds corresponding to the three unsteady branches are exactly recovered by the QST which predicts $Re_c \approx 114.7$ for the $S_1$ branch and $Re_c \approx 177.5$ and $Re_c \approx 300$ for the $F_1$ and $F_2$ branches, respectively.

Let us now comment on the spatial structure of a few modes corresponding to points $P_1$ to $P_4$ in Figure 12. Figures 13(a) and (b) suggest that the Hopf bifurcation at $P_1$ on the $F_1$ branch involves a non-negligible but moderate coupling between the disc and its wake. This moderate coupling at low-$I^*$ values is similar to what we observed with thin discs (compare with snapshots (c) and (d) in Figure 9).

Again, some variation between the real and imaginary parts of the axial velocity and vorticity disturbances at $P_2$ still on the $F_1$ branch, may be observed in snapshots (c) and (d). Although the change in the mode structure from SPT in (c) to SAT in (d) is even slightly less salient than that at $P_1$, it still suggests that this transition corresponds to a moderate interaction between the disc’s degrees of freedom and the fluid. Therefore we expect it to lead to a large-amplitude periodic motion in the saturated state, which is in line with the experimental findings of Fernandes et al. (2005) (see also Fernandes et al. (2007)) and the DNS predictions of Auguste (2010) who both considered this specific value of $I_2^*$ and observed a supercritical bifurcation from the vertical steady fall to a ZZ-type flutter. The threshold $Re_c \approx 141.0$ (at $Ar_c \approx 45.0$) and frequency $St \approx 0.11$ predicted by the LSA are in excellent agreement with the experimental ($Ar_c \approx 45$, $St \approx 0.12$) and DNS ($Ar_c \approx 44.8$, $St \approx 0.11$) observations reported in the above studies.

The spatial structure of the global mode at $P_3$ on the $S_1$ branch is displayed in snapshots (e) and (f). Since considerable changes are noticed between the two snapshots, we expect this low-frequency mode ($St \approx 0.039$) to involve a strong fluid-body coupling, certainly stronger than that at $P_1$ and $P_2$. This contrasts with the behaviour of the thin
discs considered so far, for which a low-frequency regime was only encountered on the $S_1$ branch in the large-$I^*$ range where the $St \propto I^{*-1/2}$ relation holds. In those cases, the unstable modes correspond to the quasi-static limit and hence induce only weak interactions between the body and its wake (e.g., Figures 8(i)-(j) and 10(i)-(j)). What is qualitatively similar to our previous observations with thin discs and thus emerges as a general rule is that the strength of the fluid-body coupling reaches its maximum in the moderate-$I^*$ range, to which $P_3$ belongs. That such a strong coupling, thus presumably leading to large-amplitude saturated oscillating motions with a Strouhal number as low as 0.04, may exist for a disc with $\chi = 3$ is not unlikely since planar ZZ regimes with Strouhal numbers in the range $0.025 - 0.045$ have been observed with light falling spheres, both in DNS (Jenny et al. 2004) and in experiments (Veldhuis & Biesheuvel 2007). Hence these low-frequency, yet with strong coupling, oscillating regimes seem to be specific to thick bodies. Last, figures 10(g) and (h) display the unstable mode at $P_4$ along the $F_2$ branch. Although the large-$I^*$ limit is not yet reached at this point, this mode behaves as if it were
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Figure 13: Global modes along the various stability branches of a thick disc with \( \chi = 3 \). Same convention as in Figure 8. The four investigated points have the following \((I^*, Re)\) coordinates: \( P_1(1.9 \cdot 10^{-4}, 116.7), P_2(1.6 \cdot 10^{-2}, 141), P_3(2.3 \cdot 10^{-1}, 342), P_4(5.6 \cdot 10^{-1}, 298)\).

already the case: its structure being entirely made of SAT disturbances, it belongs to the ‘fluid’ type and barely affects the disc, as confirmed by the large values reached by the normalized vorticity disturbance.

6. Final discussion and conclusions

In this study, we have considered the path instability of a disc of arbitrary thickness rising or falling in a viscous fluid due to buoyancy/gravity in the framework of a global linear analysis. Three specific configurations corresponding to discs of aspect ratio \( \chi = \infty, 10 \) and 3 have successively been examined, the latter two been thought of as prototypes of thin and thick axisymmetric bodies, respectively. Using the axisymmetric flow past a disc moving broadside on along a vertical path as the base flow, we showed existence at any value of the inertia ratio \( I^* \) of four critical global modes with an azimuthal wavenumber \( |m| = 1 \). Three of them occur through a Hopf bifurcation while the fourth is associated with a stationary (pitchfork) bifurcation. The stability diagrams in the \((I^*, Re)\) and \((I^*, St)\) planes revealed rich and non-trivial behaviours, including several points corresponding to a codimension-two bifurcation, frequency jumps along the most critical curve and local regions where, for increasing Reynolds numbers, a restabilization can follow a destabilization.

The LSA results have been systematically compared with those from DNS and experiments when available. We showed that they agree quantitatively well with previous find-
Figure 14: Superposition of (a) the critical curves and (b) the corresponding frequencies for the three discs considered in this study: $\chi = \infty$ (dotted line, red online), $\chi = 10$ (thin solid line, black online) and $\chi = 3$ (dash-dotted line, blue online). The thick portions of the lines for $\chi = \infty$ and $\chi = 10$ indicate the restabilization branches; the arrows in (b) mark the frequency jumps.

ings, both on the thresholds and frequencies, especially regarding the primary destabilization. Qualitatively, the spatial structure of the global modes normalized by the disc’s inclination angle and visualized through its real and imaginary parts, made it possible to assess qualitatively the strength of the fluid-body coupling. We found that modes involving a moderate-to-strong (resp. weak) coupling generally induce large- (resp. small-) amplitude displacements of the disc in the saturated regime. Although this statement may appear quite strong at first glance, it was proved to be robust since our inferences based on LSA results match DNS predictions remarkably well. The transitions with a non negligible mutual coupling were observed to have a common linear signature, namely a clear variation of the arrangement of axial velocity disturbances in the wake between the two different instants of time respectively associated with the real and imaginary parts of the corresponding mode.
In agreement with the recent weakly nonlinear analysis of Fabre et al. (2012), we found that the stationary mode predicted by the LSA in the freely-moving disc problem, which exhibits an $I^*$-independent threshold, differs from its counterpart for a fixed disc, even in the large-$I^*$ limit. It is worth noting that, still in the large-$I^*$ limit, this mode is not the first to be destabilized whatever the disc aspect ratio. This contrasts with the fixed-body problem in which the wake always loses its axisymmetry through the stationary bifurcation. Again in the limit of large inertia ratios, LSA predicts existence of two unstable oscillating modes which are nothing but those associated with the linear wake instability past a fixed disc. These modes have been found to involve a negligible fluid-body coupling, the wake structure being independent of the disc’s inclination, which led us to qualify them as “fluid”. The general agreement between LSA predictions and DNS results proves unambiguously that the non-vertical regimes, be the steady oblique (SO) or zigzag (either ZZ or ZZ$_2$), do not merely result from the dynamics of the sole wake but are intrinsic features resulting from the fully coupled fluid+disc problem, even though the coupling may in some cases be weak.

Finally, this study shed light on the crucial influence of the disc aspect ratio. This influence is summarized in Figure 14 which gathers the critical curves and associated frequencies obtained for the three aspect ratios we considered. Figure 14(a) reveals that for $I^* \lesssim 2 \cdot 10^{-2}$, the thick disc with $\chi = 3$ deviates from a steady vertical path at a lower Reynolds number than the thin disc with $\chi = 10$, the infinitely thin disc being the most unstable of the three. In contrast, for larger inertia ratios, the primary instability threshold is a monotonically decreasing function of the aspect ratio, the system becoming unstable through a low-frequency mode in all cases. This decrease of $Re_c$ as $\chi$ increases may be physically understood by noting that the larger $\chi$ the larger the amount of fluid that must be displaced by the body to move edgewise. Hence, for given $Re$ and $I^*$ and a given disturbance in the disc’s wake, one expects that the thicker the disc the less disturbed its motion. Thus thick discs require larger wake disturbances, i.e. a larger Reynolds number, to start moving edgewise. Figure 14(b) shows that the oscillation frequency at the threshold is insensitive to the aspect ratio for $I^* \gtrsim 1$ since, in agreement with the scaling predicted by the QST, all curves collapse on the master curve $St \approx 5.8 \cdot 10^{-2} I^{*-1/2}$. According to (D 1), the existence of this master curve implies that, for a large enough relative inertia, the variation of the dimensionless torque experienced by the disc with respect to a change in its inclination does not depend on $\chi$. The QST approximation also predicts that the threshold, $Re_c$, does not depend on $I^*$. However, as the comparison of the two subfigures shows, the latter prediction holds over a more narrow range of $I^*$ than the above prediction for the $St - I^*$ relation, a trend already noticed with two-dimensional plates (see figure 6 of Assemat et al. (2012)).

The transition scenario was found to be much more complex for low-inertia discs with two key features deserving further comments. One of these is that, although a disc with $\chi = 10$ may be thought of as thin, its behaviour still differs from that of an infinitely thin disc. In particular, the first unstable mode of the former in the low-$I^*$ limit corresponds to a steady oblique (SO) path while that of the latter results in a small-amplitude ZZ$_2$ fluttering motion. The other is that both of these modes induce bare lateral displacements of the disc, in contrast with the first unstable mode obtained for a thick disc with $\chi = 3$ which, still in the low-inertia limit, was observed to yield a ZZ regime corresponding to large-amplitude edgewise motions.

Differences between ‘thin’ and ‘very thin’ discs were already discussed in Auguste et al. (2013) and there is not much we can add here. In brief, we just remind that there are several indications that the base flow and the dynamic response of the disc+fluid system
still vary significantly with the aspect ratio for $\chi \geq 10$ because (i) vorticity production at the edge of the disc gets more intense, thus decreasing the threshold of wake instability by nearly 10\% from $\chi = 10$ to $\chi = \infty$; (ii) the drag on a disc translating edgewise at a given Reynolds number goes on decreasing with $\chi$ by nearly 15\% in that range; (iii) the mass of fluid opposing the disc’s translational or rotational accelerations (associated with the so-called added-mass effects) is also roughly 15\% less for $\chi = \infty$ than for $\chi = 10$ (Loewenberg 1993) and goes to zero for edgewise accelerations. All three features combine to make the disc accelerations and edgewise translations easier as $\chi \to \infty$ and increase the instability of the whole system. From a physical point of view, the key reason we see for this influence of $\chi$ for already ‘thin’ discs is the 180° bending of the streamlines associated with flow disturbances around the edge: the thinner the body, the larger the local curvature of these streamlines. It would be of interest to determine beyond which aspect ratio the critical curve in Figure 14 does not change significantly. We did not explore this question here. Nevertheless we may remind that the weakly nonlinear analysis of Fabre et al. (2012) showed that the stationary bifurcation from the base axisymmetric regime to the $SO$ regime switches from supercritical to subcritical for $\chi \approx 52$. Hence we expect the critical curve to remain of the same type as that corresponding to $\chi = 10$ in Figure 14 for aspect ratios of several tens, until it eventually becomes qualitatively similar to that obtained with $\chi = \infty$.

Regarding the contrasts in the magnitude of the body displacements for thin and thick discs in the low-inertia limit, we believe that it essentially finds its roots in the way the added-mass loads vary with $\chi$. This may be appreciated by splitting the total hydrodynamic force and torque in (2.13)-(2.14) into added-mass contributions resulting from the non-penetration condition of the fluid at the body surface, and vortical contributions keeping account of all viscous and wake effects (Howe 1995; Mongin & Magnaudet 2002a; Magnaudet 2011). Owing to the vanishing of the added-mass coefficient associated with edgewise translations in the limit $\chi \to \infty$, it may be shown that, for $I^* \to 0$, the in-plane projection of the force balance (2.13) implies that the lateral component of the vortical force drives the inclination of the body but has no effect on its lateral drift (Fernandes et al. 2008), the latter being then entirely regulated by the torque balance through the so-called restoring added-mass torque. In contrast, when the two translational added-mass coefficients have a similar order of magnitude as is the case for $O(1)$ aspect ratios, the body primarily reacts to a lateral force disturbance through an edgewise acceleration, its inclination then being a direct consequence of the torque disturbance. Performing a perturbative analysis of (2.13)-(2.14) with $I^* = 0$ reveals that, for a given set of lateral force and torque disturbances, the lateral velocity of the body increases with its thickness. In other words, the decrease of the in-plane added-mass coefficient as $\chi \to \infty$ restricts the ability of thin bodies with negligible inertia to perform significant lateral oscillations, thus favoring the emergence of small amplitude $A$-regimes.

Although inspection of the structure of the eigenmodes allows, to a certain extent, to disentangle the ‘fluid’ and ‘solid’ contributions to the dynamics of the system in the vicinity of the primary thresholds, predicting the super-/subcritical nature of the transitions and the amplitudes of the saturated states remain open questions which cannot be answered by linear theory. Thus, the next step of this work will be its extension to the weakly nonlinear regime, in the spirit of Meliga et al. (2009a) for fixed bodies and Fabre et al. (2012) for oblique steady paths (corresponding to the present branch $S_2$). This extension should also allow us to consider processes of modes interactions in the vicinity of the codimension-two points found at the crossings of primary branches for the three aspect ratios considered here. Such modes interactions are thought to be responsible for
some complex features observed in DNS, especially for the existence of low-amplitude ‘A-regimes’ for thin discs.

Another limitation of the present approach is the choice of the broadwise vertical fall as the base state. Study of the stability properties of the edgewise fall could also provide interesting results, especially for thick discs. Indeed, both broadwise and edgewise motions were observed to be stable through DNS (Auguste 2010) for discs with $\chi = 1$. Both extensions will require performing the LSA in a fully three-dimensional framework, which is numerically quite challenging.

**Acknowledgements**

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**Appendix A. Grid sensitivity**

We already discussed the sensitivity of the computational approach used to solve the generalized eigenvalue problem to grid characteristics in previous papers (Assemat et al. 2012; Tchoufag et al. 2013). Here we present some additional tests performed to select the grid employed in the present study. Several grids were tested by varying successively the position of the inlet ($L_1$), outlet ($L_2$) and top ($H$) of the domain with respect to the body, as well as the number $N_t$ of triangles involved in the discretization. Table 2 displays the variations of some quantities of interest when considering the instability past an infinitely thin disc with $R = 4 \cdot 10^{-4}$; the corresponding growth rates and frequencies are displayed in Figure 4. The drag coefficient $C_D$ in the base flow right at the threshold of the first instability, $Re_c = 117$ (corresponding to Figure 3) and the imaginary part of the three complex eigenvalues $S_1$, $F_1$ and $F_2$ slightly above the corresponding threshold are found to be almost insensitive to the variations of grid parameters in the range explored in Table 2. The growth rate reveals some variations, especially regarding modes $S_1$ and $F_2$. We combined these results with a similar determination of the (negative) growth rates just below the thresholds. Based on these various results, we concluded that Grids 2 and 3 are slightly less accurate than the other three and we finally selected Grid 1 as the best compromise between accuracy and computational time. All results discussed in the paper were obtained with that grid.

<table>
<thead>
<tr>
<th>Grid</th>
<th>$L_2$</th>
<th>$L_1$</th>
<th>$H$</th>
<th>$N_t$</th>
<th>$C_D(Re = 117)$</th>
<th>$\lambda_{S_1}$</th>
<th>$\lambda_{F_1}$</th>
<th>$\lambda_{S_2}$</th>
<th>$\lambda_{F_2}$</th>
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<tr>
<td>G1</td>
<td>76</td>
<td>23</td>
<td>33</td>
<td>20693</td>
<td>1.197</td>
<td>2.469 - $10^{-3}$ ± 1.573i</td>
<td>6.264 - $10^{-3}$ ± 0.667i</td>
<td>3.395 - $10^{-3}$</td>
<td>4.580 - $10^{-3}$ ± 1.248i</td>
</tr>
<tr>
<td>G2</td>
<td>101</td>
<td>23</td>
<td>33</td>
<td>22091</td>
<td>1.196</td>
<td>2.804 - $10^{-3}$ ± 1.571i</td>
<td>6.373 - $10^{-3}$ ± 0.667i</td>
<td>3.417 - $10^{-3}$</td>
<td>5.030 - $10^{-3}$ ± 1.248i</td>
</tr>
<tr>
<td>G3</td>
<td>76</td>
<td>44</td>
<td>33</td>
<td>21963</td>
<td>1.196</td>
<td>2.618 - $10^{-3}$ ± 1.572i</td>
<td>6.259 - $10^{-3}$ ± 0.667i</td>
<td>3.410 - $10^{-3}$</td>
<td>4.862 - $10^{-3}$ ± 1.248i</td>
</tr>
<tr>
<td>G4</td>
<td>76</td>
<td>55</td>
<td>33</td>
<td>23849</td>
<td>1.197</td>
<td>2.216 - $10^{-3}$ ± 1.572i</td>
<td>6.252 - $10^{-3}$ ± 0.667i</td>
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<td>4.746 - $10^{-3}$ ± 1.248i</td>
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<td>1.197</td>
<td>2.221 - $10^{-3}$ ± 1.572i</td>
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<td>3.526 - $10^{-3}$</td>
<td>4.846 - $10^{-4}$ ± 1.248i</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Influence of grid characteristics on some selected quantities of interest. The values of Reynolds number of modes $S_1$, $F_1$, $S_2$ and $F_2$ are $Re \approx 106, 114, 143$ and 168, respectively.
Appendix B. The generalized eigenproblem for the coupled fluid-body system

The set of linear equations to be solved is given by (2.11)-(2.15). Seeking solutions in the form of normal modes (2.16) and defining \( I^{**} \) as \( I^*/(1 + \frac{3}{4} \lambda^{-2}) \) yields

\[
\lambda \mathbf{v} = - (\mathbf{V}_0 - \mathbf{U}_0) \cdot \nabla_m \mathbf{v} - (\mathbf{v} - (\hat{\omega} + \mathbf{r}) e^{-im\varphi}) \cdot \nabla \mathbf{V}_0 - \hat{\omega} e^{-im\varphi} \times \mathbf{V}_0 - \nabla_m \hat{p} + Re^{-1} \nabla_m^2 \hat{v},
\]

\[
0 = \nabla_m \cdot \mathbf{v},
\]

\[
16 \lambda I^{**} \mathbf{u} = -16 I^{**} \hat{\omega} \times \mathbf{U}_0 + D_0 (\hat{\theta}_r \mathbf{y} - \hat{\theta}_x \mathbf{z})
\]

\[
+ \oint_{\varphi} \left[ \hat{f}_r \mathbf{x} + (\hat{f}_r \cos \varphi - \hat{f}_\varphi \sin \varphi) \mathbf{y} + (\hat{f}_r \sin \varphi + \hat{f}_\varphi \cos \varphi) \mathbf{z} \right] e^{im\varphi} dS,
\]

\[
\lambda I^{**} \cdot \mathbf{w} = \oint_{\varphi} \left[ r \hat{f}_r \mathbf{x} + \left( (r \hat{f}_r - x \hat{f}_x) \sin \varphi - x \hat{f}_\varphi \cos \varphi \right) \mathbf{y} + \left( (x \hat{f}_r - r \hat{f}_x) \cos \varphi - x \hat{f}_\varphi \sin \varphi \right) \mathbf{z} \right] e^{im\varphi} dS,
\]

\[
\lambda \mathbf{\xi} = \mathbf{\hat{\omega}}.
\]

where \( \nabla_m = (\partial_r, \frac{im}{r}, \partial_\varphi) \) and \((\hat{f}_r, \hat{f}_\varphi, \hat{f}_x)^T = \mathbf{i}_m \cdot \mathbf{n}, \mathbf{i}_m = -\mathbf{p} I + Re^{-1} (\nabla_m \hat{v} + (\nabla_m \hat{v})^T) \) being the disturbance stress tensor relative to mode \( m \).

The boundary conditions to be satisfied by the fluid components of the problem on the symmetry axis \( \mathcal{S} \) depend on the value of the azimuthal wavenumber. After some inspection one finds

- for \( m = 0 \): \( \hat{v}_r = \partial_r \hat{v}_x = \partial_r \hat{p} = 0 \),
- for \( |m| = 1 \): \( \partial_r \hat{v}_r = \partial_r \hat{v}_x = \hat{v}_x = \hat{p} = 0 \),
- for \( |m| \geq 2 \): \( \hat{v}_r = \hat{v}_\varphi = \hat{v}_x = \hat{p} = 0 \).

Finally, projecting (B3)-(B5) onto each axis of the moving frame of reference \((x, y, z)\) yields different sets of rigid-body motion equations, depending on the value of \( m \) under consideration. These equations are detailed in the following three subsections.

B.1. Case \( m = 0 \)

Since this mode preserves axial symmetry (possibly with swirl), only \( \hat{u}_x \) and \( \hat{w}_x \) can be nonzero provided \( \lambda \neq 0 \). Noting that \( \hat{n}_x \) is nonzero (actually \( \hat{n}_x = \pm 1 \)) only on the front and back faces of the disc where \( dS = r dr d\varphi \), while \( \hat{n}_r \) is nonzero (actually \( \hat{n}_r = 1 \)) only on its lateral surface where \( dS = \frac{1}{2} r dr d\varphi \), the eigenproblem reduces to

\[
8 I^{**} \lambda \hat{u}_x = \pi \int_{\varphi} \left[ (-\hat{\rho} + 2Re^{-1} \partial_x \hat{\varphi}) n_x r dr + \frac{1}{2} Re^{-1} (\partial_x \hat{\varphi} + \partial_x \hat{v}_r) n_x dx \right],
\]

\[
\lambda I^{**} \hat{w}_x = \pi Re^{-1} \int_{\varphi} \left[ (\partial_x \hat{v}_r) n_x r^2 dr + \frac{1}{2} (\partial_x \hat{v}_t - \frac{\hat{\varphi}}{r}) n_x dx \right].
\]

This problem may be recast in the generic form

\[
\lambda \mathcal{B}_0 \hat{q}_0 = \mathcal{A}_0 \hat{q}_0
\]

by defining

\[
\mathcal{A}_0 = \begin{pmatrix}
-\mathcal{C}_0(\cdot, \mathbf{V}_0) + Re^{-1} \nabla_0^2(\cdot) & -\nabla_0(\cdot) & \partial_x (V_0 e_r + V_0 x) & -V_0 e_r \\
\nabla_0(\cdot) & 0 & 0 & 0 \\
\mathcal{F}(\cdot) & 0 & 0 & 0 \\
\mathcal{M}^*(\cdot) & 0 & 0 & 0
\end{pmatrix},
\]

(9)
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\[ \mathbf{B}_0 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 8 \mathbf{I}^{**} & 0 \\ 0 & 0 & 0 & \mathbf{I}^{**} \end{pmatrix} \quad \text{and} \quad \mathbf{q}_0 = \begin{pmatrix} \dot{\psi} \\ \dot{\psi} \\ \dot{u}_x \\ \dot{\omega}_x \end{pmatrix}, \quad (B10) \]

with \( \mathcal{C}_m(\dot{v}, \mathbf{V}_0) = (\mathbf{V}_0 - \mathbf{U}_0) \cdot \nabla_m \dot{v} + \dot{v} \cdot \nabla \mathbf{V}_0 \), \( \mathcal{F}^v \) (resp. \( \mathcal{M}^v \)) and \( \mathcal{F}^p \) being the operators that generate the viscous and pressure contributions to the hydrodynamic force (resp. torque). Here \( \mathcal{F}^v(\mathbf{V}_0) \) and \( \mathcal{F}^p(\mathbf{p}_0) \) correspond to the right-hand side of (B6), while \( \mathcal{M}^v(\mathbf{V}_0) \) corresponds to that of (B7). In the eigenproblem (B8), the boundary conditions are imposed through a penalization method. For instance, the no-slip condition on the disc surface is satisfied by inserting the equation \( \varepsilon^{-1} [\dot{v} - \dot{u}_x \mathbf{x} - \dot{r}_x \mathbf{x} - \dot{\omega}_x \mathbf{e}_y] = 0 \) with \( \varepsilon \ll 1 \) in the rows of the matrices \( \mathcal{A}_0 \) and \( \mathcal{B}_0 \) corresponding to boundary nodes, i.e., by replacing the first row of \( \mathcal{A}_0 \) (resp. \( \mathcal{B}_0 \)) with \( (\varepsilon^{-1}, 0, -\varepsilon^{-1} \mathbf{x}, -\varepsilon^{-1} \mathbf{e}_y) \) (resp. 0). In this way, the boundary condition is satisfied if \( \varepsilon \) is selected small enough for \( O(\varepsilon^{-1}) \) terms to dominate all other terms at these nodes.

\section*{B.2. Case \( m = \pm 1 \)}

In this mode, \( \dot{u}_x = \dot{\omega}_x = 0 \) provided \( \mu \neq 0 \), so that

\[
16\lambda \mathbf{I}^{**} \dot{u}_y = 16\lambda \mathbf{I}^{**} U_0 \dot{\omega}_x + D_0 \dot{\theta}_z + \pi \int_{\mathcal{S}} \frac{1}{2} (\dot{\rho} + 2\Re^{-1}\partial_r \dot{v}_r) n_r dx + \Re^{-1}(\partial_r \dot{v}_x + \partial_x \dot{v}_r) n_x r dr \\
\mp i\pi \Re^{-1} \int_{\mathcal{S}} \frac{1}{2} (\partial_r \dot{\psi} - \dot{\psi} \frac{1}{r} \dot{v}_r \pm \frac{1}{r} \dot{v}_x \pm \frac{1}{r} \dot{v}_z \pm \frac{1}{r} \dot{v}_z \pm \frac{1}{r} \dot{v}_z) n_r dx, \quad (B11)
\]

\[
16\lambda \mathbf{I}^{**} \dot{u}_z = -16\lambda \mathbf{I}^{**} U_0 \dot{\omega}_y - D_0 \dot{\theta}_y \pm i\pi \int_{\mathcal{S}} \frac{1}{2} (\dot{\rho} + 2\Re^{-1}\partial_r \dot{v}_r) n_r dx + \Re^{-1}(\partial_r \dot{v}_x + \partial_x \dot{v}_r) n_x r dr \\
\mp \pi \Re^{-1} \int_{\mathcal{S}} \frac{1}{2} (\partial_r \dot{\psi} - \dot{\psi} \frac{1}{r} \dot{v}_r \pm \frac{1}{r} \dot{v}_x \pm \frac{1}{r} \dot{v}_z \pm \frac{1}{r} \dot{v}_z \pm \frac{1}{r} \dot{v}_z) n_r dx, \quad (B12)
\]

\[
\lambda \mathbf{I}^* \dot{\omega}_y = \pm i\pi \int_{\mathcal{S}} x \frac{1}{2} (\dot{\rho} + 2\Re^{-1}\partial_r \dot{v}_r) n_x r dr + \Re^{-1}(\partial_r \dot{v}_x + \partial_x \dot{v}_r) n_x dx \\
\mp i\pi \int_{\mathcal{S}} x \frac{1}{2} (\dot{\psi} - \dot{\psi} \frac{1}{r} \dot{v}_r \pm \frac{1}{r} \dot{v}_x \pm \frac{1}{r} \dot{v}_z \pm \frac{1}{r} \dot{v}_z \pm \frac{1}{r} \dot{v}_z) n_x r dr, \quad (B13)
\]

\[
\lambda \mathbf{I}^* \dot{\omega}_z = -\pi \int_{\mathcal{S}} x \frac{1}{2} (\dot{\rho} + 2\Re^{-1}\partial_r \dot{v}_r) n_x r dr + \Re^{-1}(\partial_r \dot{v}_x + \partial_x \dot{v}_r) n_x dx \\
\mp \pi \int_{\mathcal{S}} x \frac{1}{2} (\dot{\psi} - \dot{\psi} \frac{1}{r} \dot{v}_r \pm \frac{1}{r} \dot{v}_x \pm \frac{1}{r} \dot{v}_z \pm \frac{1}{r} \dot{v}_z \pm \frac{1}{r} \dot{v}_z) n_x r dr, \quad (B14)
\]

\[
\lambda \dot{\theta}_y = \dot{\omega}_y, \quad (B15)
\]

\[
\lambda \dot{\theta}_z = \dot{\omega}_z. \quad (B16)
\]

Using the U(1)-coordinate transformation \( \dot{u}_\pm = \dot{u}_y \mp i \dot{u}_z, \dot{\theta}_\pm = \dot{\theta}_y \pm i \dot{\theta}_z \) and \( \dot{\omega}_\pm = \dot{\omega}_x \mp i \dot{\omega}_y \), the \( y \) and \( z \) projections can be added so as to reduce the body equations for helical disturbances to
\[16\lambda^{*}\hat{u}_{\pm} = \pm 16iI^{*}U_{0}\hat{\omega}_{\pm} \pm iD_0\hat{\theta}_{\pm} + 2\pi \int_{\gamma} \left[ \frac{1}{2}(-\hat{p} + 2Re^{-1}\partial_{\hat{r}}\hat{v}_{\lambda})n_{r}dx + Re^{-1}(\partial_{\hat{r}}\hat{v}_{\lambda} + \partial_{\hat{z}}\hat{v}_{\lambda})n_{r}rd\gamma \right]
\]
\[\mp 2\pi Re^{-1} \int_{\gamma} \left[ \frac{1}{2}(\partial_{\hat{r}}\hat{v}_{\lambda} - \frac{\hat{v}_{\lambda}}{r})n_{r}dx + (\partial_{\hat{r}}\hat{v}_{\lambda} \pm \frac{1}{r}\hat{v}_{\lambda})n_{r}rd\gamma \right] , \quad (B \, 17)\]
\[\lambda^{*}\hat{\omega}_{\pm} = \mp 2\pi \int_{\gamma} r \left[ (-\hat{p} + 2Re^{-1}\partial_{\hat{r}}\hat{v}_{\lambda})n_{r}rd\gamma + \frac{1}{2}Re^{-1}(\partial_{\hat{r}}\hat{v}_{\lambda} + \partial_{\hat{z}}\hat{v}_{\lambda})n_{r}dx \right]
\]
\[\pm 2\pi \int_{\gamma} x \left[ \frac{1}{2}(-\hat{p} + 2Re^{-1}\partial_{\hat{r}}\hat{v}_{\lambda})n_{r}dx + Re^{-1}(\partial_{\hat{r}}\hat{v}_{\lambda} + \partial_{\hat{z}}\hat{v}_{\lambda})n_{r}rd\gamma \right]
\]
\[-2\pi \int_{\gamma} x \left[ \frac{1}{2}Re^{-1}(\partial_{\hat{r}}i\hat{v}_{\lambda} - \frac{i\hat{v}_{\lambda}}{r} \mp \hat{v}_{\lambda})n_{r}dx + Re^{-1}(\partial_{\hat{r}}i\hat{v}_{\lambda} \mp \hat{v}_{\lambda})n_{r}rd\gamma \right] , \quad (B \, 18)\]
\[\lambda\hat{\theta}_{\pm} = \hat{\omega}_{\pm} . \quad (B \, 19)\]

Comparing (B18) with (B13)-(B14) shows that \(\hat{\omega}_{\pm} = 2\hat{\omega}_{\pm} = \pm 2\hat{\omega}_{\pm}\), so that (B19) and (B15)-(B16) imply \(\hat{\theta}_{\pm} = 2\hat{\theta}_{\pm} = \pm 2\hat{\theta}_{\pm}\). Introducing these results into (B11)-(B12) and comparing with (B17) finally shows that \(\hat{u}_{\pm} = 2\hat{u}_{\pm} = \pm 2\hat{u}_{\pm}\). This could have been inferred from the fact that the transformation \((\hat{u}_{\gamma}, \hat{\omega}_{\pm}, \hat{\theta}_{\pm}) \rightarrow \pm (\hat{u}_{\gamma}, \hat{\omega}_{\pm}, \hat{\theta}_{\pm})\) for \(m = \pm 1\) interchanges the \(y\)- and \(z\)-projections. Therefore the stability analysis could have been restricted to either the \(y\)- or the \(z\)-direction. Taking the above relations into consideration and introducing the U(1)-coordinate transformation in the fluid equations, the eigenproblem may finally be written in the form \(\lambda \mathcal{B}_{\pm 1}\mathbf{q}_{\pm 1} = \mathcal{A}_{\pm 1}\mathbf{q}_{\pm 1}\) \((B \, 20)\) with
\[
\mathcal{A}_{\pm 1} = \begin{pmatrix}
-\mathcal{C}_1(\cdot, \mathbf{V}_0) + Re^{-1}\nabla^2_1(\cdot) & -\nabla_1(\cdot) & \frac{1}{2}\partial_r(V_0\mathbf{e}_r + V_0\mathbf{x}) & \mp \frac{1}{2}i\partial_r(V_0\mathbf{e}_r + V_0\mathbf{x}) - (V_0\mathbf{e}_r + V_0\mathbf{x}) \quad 0

\nabla_1(\cdot) & 0 & 0 & 0

\mathcal{F}(\cdot) & \mathcal{F}^p(\cdot) & 0 & 0

\mathcal{M}(\cdot) & \mathcal{M}^p(\cdot) & 0 & 0

\end{pmatrix},
\]
\[
\mathcal{B}_{\pm 1} = \begin{pmatrix}
\mathbf{I} & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0

0 & 0 & 16\lambda^{*}U_0 & 0 & D_0

0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
and \(\mathbf{q}_{\pm 1} = \begin{pmatrix}
\hat{\psi} \\
\hat{\rho} \\
\hat{\omega}_{\pm} \\
\hat{\theta}_{\pm}
\end{pmatrix}\).

where \(\mathcal{F}(\hat{\psi}_1)\) and \(\mathcal{F}^p(\hat{\psi}_1)\) (resp. \(\mathcal{M}(\hat{\theta}_1)\) and \(\mathcal{M}^p(\hat{\theta}_1)\)) correspond to the right-hand side of (B17) (resp. (B18)).

B.3. Case \(|m| \geq 2\)

For perturbations of larger azimuthal wavenumbers, it is straightforward to see that there is no more coupling between the fluid and body equations. As (B4) shows, the torque exerted by the fluid on the disc is zero, so that \(\hat{\omega} = \mathbf{0}\) for any \(\lambda \neq 0\), which by virtue of (B5) implies \(\hat{\mathbf{z}} = \mathbf{0}\) under the same condition. The same is of course also true for the hydrodynamics force in (B3), so that \(\hat{\mathbf{u}} = \mathbf{0}\).
Hence the formerly coupled eigenproblem reduces to its fluid component, namely

$$\lambda \mathcal{B}_m \mathcal{q}_m = \mathcal{A}_m \mathcal{q}_m$$  \hspace{1cm} (B 21)

with

$$\mathcal{A}_m = \begin{pmatrix} -\mathcal{G}_m(\cdot, \mathbf{v}_0) - \frac{1}{Re} \nabla \mathcal{G}_m(\cdot) & -\mathcal{V}_m(\cdot) \\ \nabla \mathcal{G}_m(\cdot) & 0 \end{pmatrix}, \quad \mathcal{B}_m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\mathcal{q}}_m = \begin{pmatrix} \hat{\mathbf{v}}_r \\ \hat{\mathbf{v}}_\theta \end{pmatrix}. \hspace{1cm} (B 22)$$

**Appendix C. The $m = 0$ and $m = 2$ modes**

In this appendix we comment on the thresholds and spatial structures of the $m = 0$ and $m = 2$ modes. Although these two families cannot lead to path instability for reasons discussed in section 2, some of them may be of importance in future weakly nonlinear studies.

**C.1. $m = 0$**

To illustrate the behaviour and structure of the $m = 0$ modes, we solve (B 8) in the case of an infinitely thin disc. We select two values of the inertia ratio corresponding to points $P_1$ ($I^* = 5 \cdot 10^{-3}$) and $P_2$ ($I^* = 5 \cdot 10^1$) in Figure 5. Three axisymmetric global modes are observed. They are all found to be stationary ($St = 0$) and stable as illustrated by their negative growth rates displayed in Figure 15, even though two of them are almost neutral in the limit $I^* \to \infty$ (see the inset in Figure 15).

From the structure of (B 7), (B 9) and (B 10), one can separate the axisymmetric modes into two families: $U_{RX}$ and $U_{PR}$. The first (resp. second) family is made of modes whose velocity components lie along $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ (resp. $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$) and thus yield a purely azimuthal (resp. axial and radial) vorticity field. On this basis we find that, whatever the inertia ratio, two series of modes belong to $U_{RX}$ and only one belongs to $U_{PR}$. Their structure at $Re = 328$ is displayed in Figure 16 where, for obvious reasons, $U_{RX}$ (resp. $U_{PR}$) modes have been normalized by $\hat{u}_x$ (resp. $\hat{\omega}_x$).
Figure 16: Axisymmetric modes of an infinitely thin disc at $Re = 328$ for $I^* = 4 \cdot 10^{-3}$ (left) and $I^* = 5 \cdot 10^{-3}$ (right). For both values of $I^*$, the stability of the global mode increases from top to bottom. Modes (a), (d), (e), (f) belong to the $U_{RX}$ family and are normalized such that $\hat{u}_x = 1$, while those in (b) and (c) belong to the $U_P$ family and are normalized such that $\hat{\omega}_x = 1$. For $U_{RX}$ (resp. $U_P$) modes, the upper half of each snapshot displays the axial (resp. azimuthal) velocity while the lower half displays the azimuthal (resp. axial) vorticity.

The large isovales of the axial velocity and azimuthal vorticity observed for the $U_{RX}$ mode displayed in snapshot (f) indicate that the corresponding mode has a ‘fluid’ nature, i.e. it could also have been obtained by studying the wake of a fixed disc (this is why the corresponding curve is labelled ‘F’ in Figure 15). In contrast the two almost neutral modes displayed in snapshots (b) and (d) are tight to the body degrees of freedom and could also be retrieved by using the quasi-static model summarized in Appendix D. Their small growth rates result from the $|\lambda_r| \sim I^{*-1}$ relationship (D 2). The mode displayed in snapshot (d) was also observed with two-dimensional plates and rods by Fabre et al. (2011). It was termed ‘Back-to-Terminal-Velocity’ (BTV) because it tends to dampen any perturbation that changes the relative velocity between the body and fluid in the base flow. For a similar reason, the mode displayed in snapshot (b) which is of $U_P$-type can be termed ‘Back-to-Zero-Rotation rate’ (BZR). These two terminologies are used in the inset of Figure 15 where the growth rate of the corresponding modes is shown. While past studies tend to ignore the role of axisymmetric modes in the dynamics of bluff bodies wakes, the BTV and BZR modes reported here deserve some attention. Indeed, being very weakly damped, they may influence the nonlinear evolution of the system by interacting with the unstable $LF$ mode at point $P5$ of Figure 5 (see Figure 8(g)-(h)). At low inertia ratios, the distinction between ‘fluid’ and ‘solid’ (or ‘aerodynamic’) modes generally makes no sense. At first glance the structure of the mode displayed in snapshot (a) looks very similar to that in (f), suggesting that the ‘F’ branch exists for all $I^*$. However, the mode in (a) cannot be considered as purely ‘fluid’ since the corresponding isovales of the axial velocity and azimuthal vorticity indicate a non-negligible coupling between the fluid and the disc’s degrees of freedom. Snapshots (c) and (e) reveal spatial structures very different from those of snapshots (b) and (d). Hence one has to conclude...
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Table 3: Thresholds and frequencies of the most unstable global modes of azimuthal wavenumber $|m| = 2$.

![Figure 17](image)

Figure 17: The first three unstable global modes $|m| = 2$ in the wake of a disc with $\chi \to \infty$. The streamwise velocity (resp. vorticity) is shown in the upper (resp. lower) part of each subfigure; each mode is normalized in such a way that the stationary eigenvector written in the form $(\hat{v}_r, i\hat{v}_\phi, \hat{v}_x, \hat{p})$ is real.

that the aforementioned BZR and BTV modes ‘disappear’ at low enough $I^*$. This is not unlikely since, according to Figure 15, the two branches corresponding to snapshots (c) and (e) are much more damped than the branches BTV and BZR observed at large $I^*$.

C.2. $m = 2$

Since the eigenproblem reduces to its fluid component for $|m| \geq 2$, the threshold of the corresponding unstable modes is independent of the body-to-fluid inertia ratio and is identical to the threshold of the wake instability that would be observed, were the body held fixed.

Table 3 displays the instability thresholds of the most unstable $|m| = 2$ modes for each of the three body aspect ratios investigated throughout this paper. As shown in the table, it turns out that, in contrast to the case $m = 1$, the first bifurcation is of Hopf type and is followed by a stationary bifurcation. Figure 17 finally displays the structure of the first three global modes for an infinitely thin disc.
Appendix D. Summary of the quasi-steady theory (QST)

The QST presented by Fabre et al. (2011) was derived through a rigorous asymptotic expansion to predict analytically the instability characteristics of a freely falling two-dimensional body in the high-mass limit \( I^* \rightarrow \infty \). It is mainly based on the idea that in this limit an unsteady body motion happens on a time scale much larger than those governing the flow dynamics. This assumption allows the so-called ‘aerodynamic’ modes tied to the body to be computed with the flow considered quasi-steady. Since the present problem with \(|m| = 1\) can be reduced to an almost two-dimensional problem by using the \( U(1) \)-coordinates, the two-dimensional QST formulation still holds. Therefore, equation (15) from Fabre et al. (2011) may be applied and the most unstable eigenvalue reads at criticality

\[
\lambda_{LF} = \frac{1}{2I^*} \left( \frac{\mathcal{L}_\alpha}{16} + \mathcal{M}_\omega \right) \pm \sqrt{\frac{\mathcal{M}_\omega}{I^*}},
\]

where \( LF \) stands for ‘low frequency’, \( \mathcal{L}_\alpha \) (resp. \( \mathcal{M}_\omega \)) denotes the lift force (resp. torque) due to a change in the incidence angle (defined as the angle between the disc translational velocity and its symmetry axis) and \( \mathcal{M}_\omega \) denotes the torque induced by a weak rotation of the disc about one of its diameters. These coefficients can be computed by solving a series of elementary problems, following the procedure described by Fabre et al. (2011) for two-dimensional bodies.

When the right-hand side of (D1) is complex (i.e. \( \mathcal{M}_\omega \) is negative), searching for the value of \( Ar \) at which the real part vanishes provides the corresponding threshold; note that this threshold does not depend on \( I^* \). Fabre et al. (2011) showed that such a threshold exists for two-dimensional square rods above \( Re = 48 \), yielding the onset of an unstable oscillating mode characterized by a strong coupling between the body and its wake. A more thorough study of the QST for three-dimensional bodies will be the subject of a future paper. Here, it is enough to say that the \( S_1 \) mode observed in the present study for the three aspect ratios we considered is the counterpart of the mode observed for the aforementioned square rod, and that the conclusions of the two-dimensional study regarding this mode are globally applicable here. However, a central difference in the compared behaviour of two- and three-dimensional bodies must be pointed out: in the limit of large inertia ratios, we found the ‘solid’ \( S_1 \) mode to be more unstable than the ‘fluid’ modes for discs, while in the case of the square rod the former was found to be slightly less unstable than the latter (namely, the classical Von Kármán shedding mode which emerges at \( Re \approx 44 \)).

Last, let us mention that the QST also predicts an antisymmetric ‘back-to-vertical’ (BV) and an axisymmetric ‘back-to-terminal-velocity’ (BTV) stationary mode whose growth rates may be shown to be always negative. As discussed in Appendix C, in the present three-dimensional case there is also an axisymmetric ‘back-to-zero-rotation rate’ (BZR) global mode which is stationary and stable. The growth rates of these last two damped modes are

\[
\lambda_{BTV} = -\frac{D_{u}}{16I^{**}}, \quad \lambda_{BZR} = -\frac{\mathcal{M}_\omega}{2I^{**}},
\]

where \( D_{u} \) stands for the drag variation due to a change in the body velocity, \( \mathcal{M}_\omega \) denotes the axial torque due to a weak rotation about the disc axis, and again \( I^{**} = I^*(1 + \frac{4}{3} \chi^{-2})^{-1} \).
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REFERENCES


